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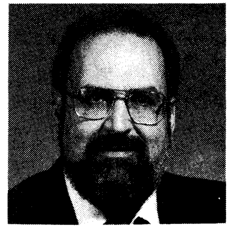
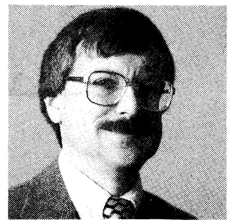
An Algebraic Approach to Some Number-Theoretic Problems Arising from Paper-Folding Regular Polygons

JON FROEMKE AND JERROLD W. GROSSMAN



JON FROEMKE: I received my Ph.D. from the University of California at Berkeley in 1967 in universal algebra under the direction of Alfred Foster. I have been at Oakland University ever since. This article represents a return to number theory, my first love, and in an unusual way, a continuation of my interest in the arithmetic of primal algebras.



JERROLD W. GROSSMAN: I specialized in mathematical logic at Stanford University (B.S., M.S. in 1970), studied algebraic topology at M.I.T. (Ph.D. in 1974 under Daniel Kan), and have been at Oakland University since 1974. My research articles have dealt with algebraic topology, algebra, graph theory, combinatorics, probability and statistics, computer science, and now, apparently, number theory.

Introduction. Since 1983 Peter Hilton and Jean Pedersen have been studying the ramifications of an ingenious paper-folding construction they devised for approximating angles and regular polygons [**2**], [**3**], [**4**], [**5**], [**6**], [**7**]. They are quickly led into number-theoretic questions. In this article we will see that with the use of some additional tools of graph theory, number theory, and algebra, we can more fully explain the phenomena they are studying. We also pose some new questions they have not raised and answer many of them. Perhaps not surprisingly, our investigations touch on several aspects of elementary number theory and lead quickly to some of its famous unsolved problems. On the other hand, Hilton and Pedersen pursue other interesting number-theoretic aspects of the construction, which we do not consider, so in no sense does our paper supersede their work.

This article is somewhat self-contained, in the sense that most of the number theory we need is reviewed herein. The underlying problems are extremely easy to state, and thus the article should be accessible to a wide audience. Indeed, the basic numerical construction proved fascinating to the first author's ten-year-old son, who could appreciate the questions being raised, but not, of course, the reasons behind their answers.

In what follows, we will let **Z** denote the set of integers, and for integers x and y we will let (x, y) and $[x, y]$ denote their greatest common divisor and least common multiple, respectively (with the obvious extensions to more than two variables).

Since the statement of the problems we are considering cannot be given until Section 2, it would be meaningless to give an outline of the paper at this point. The usual preview, section by section, is instead given at the end of Section 2.

1. Folding angles: the geometric motivation. Hilton and Pedersen have a clever method for approximating rational angles, to any desired degree of accuracy, using only an elementary paper-folding operation: bisecting an angle. (By rational angle we mean an angle whose measure is a rational number of degrees.) We summarize their procedure briefly. Although the geometric problem has no direct bearing on the rest of this paper, it serves as the motivation for considering the numerical construction of Section 2.

Let $r_0 = a\pi/b$ be a given acute rational angle, i.e., assume that $0 < a < b/2$, where a and b are integers with $(a, b) = 1$. Call b the *denominator* of r_0. For their purposes, Hilton and Pedersen assume initially that b is odd, and we will also (for now) make this assumption. Suppose angle r_0 is formed between the bottom edge of a long strip of paper and a fold in the paper (see FIGURE 1).

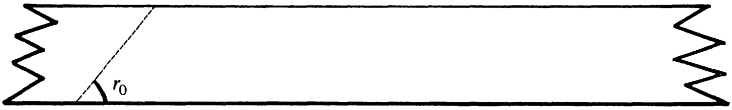


FIG. 1. Angle r_0 on a strip of paper.

If a is even, say $a = 2a'$, then by folding the original crease onto the bottom edge of the strip, we can construct angle $r_1 = a'\pi/b$ (see FIGURE 2).

FIG. 2. Bisecting $a\pi/b$ when a is even.

On the other hand, if a is odd, then the supplementary adjacent interior angle at the top of the strip has measure $(b - a)\pi/b$, with $(b - a)$ even, say equal to $2a''$. We can, therefore, bisect it by folding the original crease onto the top edge of the strip and, thereby, form $r_1 = a''\pi/b$ along the top of the strip (see FIGURE 3).

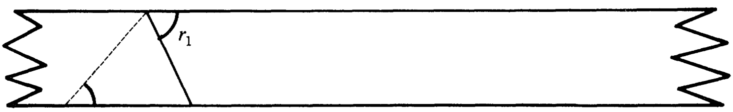


FIG. 3. Bisecting $(b - a)\pi/b$ when a is odd.

Iterating this procedure, we obtain a sequence of acute rational angles $r_0, r_1, r_2, \ldots$ with denominator b. It turns out (as a consequence of some observations in Section

2) that the sequence is cyclic, and hence that $r_0 = r_k$ for some k (to be called the quasiorder of 2 mod b). See FIGURE 4.

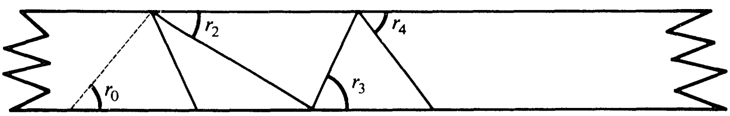


FIG. 4. $r_0 = r_4$.

Now if in fact r_0 is only an approximation to $a\pi/b$, say with a small error ε, then it is easy to see that the r_k constructed by this procedure is again an approximation to $a\pi/b$, but with error $\varepsilon/2^k$. Thus, by iterating the process, we can construct arbitrarily good approximations to $a\pi/b$.

Using this paper-folding procedure, Hilton and Pedersen devised a systematic method for constructing arbitrarily good approximations to any regular convex polygon or regular star polygon.

2. Charm bracelets: the numerical construction. Isolating the numerical ingredients of the foregoing geometric construction, we have simply the following operation on the set of positive integers less than b and relatively prime to b, where b is a fixed odd integer greater than 2:

if a is even, then halve it, i.e., let $a' = a/2$;

if a is odd, then subtract it from b, i.e., let $a' = b - a$.

(The reader may have noticed the similarity of this operation to the one that appears in the Collatz problem [9], although we have found nothing here that seems relevant to that notorious problem.) We will find it easier (but equivalent) in what follows to work instead with the opposite operation, which we now define.

DEFINITION. Let $V(b) = \{a \in \mathbf{Z} \mid 0 < a < b \text{ and } (a, b) = 1\}$. The function f_2: $V(b) \to V(b)$ is given by the rule

$$f_2(a) = \begin{cases} 2a & \text{if } a < b/2 \\ b - a & \text{if } a > b/2 \end{cases}.$$

(The subscript is 2 because of the multiplication by 2; we generalize this later.) To get a picture of this operation, we construct, for each odd $b > 1$, a graph whose vertices are the elements of $V(b)$ and whose edges are all the unordered pairs $(a, f_2(a))$. (If $b = 3$, then we put two edges between 1 and 2, since $f_2(1) = 2$ and $f_2(2) = 1$.) The graph for $b = 17$ is shown in FIGURE 5.

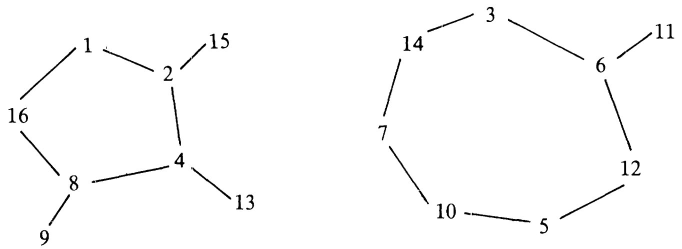


FIG. 5. The graph for $b = 17$.

The following observations about these graphs follow immediately from the definitions.

1. Each vertex a for which a is odd and greater than $b/2$ has degree 1, being adjacent only to $b - a$. Such values of a we will call *charms*.
2. Each vertex a for which a is even and less than $b/2$ has degree 3, being adjacent to $2a$, $a/2$, and the charm $b - a$.
3. Each vertex a for which a is even and greater than $b/2$ has degree 2, being adjacent to $b - a$ and $a/2$.
4. Each vertex a for which a is odd and less than $b/2$ has degree 2, being adjacent to $b - a$ and $2a$.

Now if we momentarily discard the charms, then every vertex has degree 2, and hence (by a trivial result of graph theory) the graph consists of one or more disjoint cycles (polygons). We call such a cycle, together with all the charms connected to it, a *bracelet*. We call the number of vertices in a bracelet its *weight*, and (continuing the jewelry metaphor) we call the number of noncharms in a bracelet its *size*. Finally we let $B(2, b)$ denote the number of bracelets in the graph. For $b = 17$, we see from FIGURE 5 that $B(2, 17) = 2$, that the weight of each bracelet is 8, and that the sizes of the two bracelets are 5 and 7, respectively.

Our bracelet graph replaces Hilton and Pedersen's "symbol" for representing the same information. They would display the second bracelet in FIGURE 5, for example, as

$$17\begin{vmatrix} 3 & 7 & 5 \\ 1 & 1 & 2 \end{vmatrix}.$$

This symbol is interpreted as follows: beginning with $a = 3$, we subtract a from 17 and divide the answer (14) as many times as we can by the number 2 (in this case, we divide once, obtaining the number 7 as the next value of a). The number of factors of 2 that were divided out appears in the second row, and the next value of a appears in the first row of the next column. The process is repeated until the original value of a is obtained. Two differences between our approaches are worth noting. First, Hilton and Pedersen list only the odd values of a less than $b/2$; the missing intermediate even values are implied and the second row in each column shows the number of missing even values. (Hilton and Pedersen use their symbol, among other things, to succinctly encode the instructions for paper-folding approximations to star polygons.) Second, the symbol, as read from left to right, represents divisions by 2, rather than multiplications by 2, as in our approach, although Hilton and Pedersen also present a multiplication-based, as well as a division-based, "quasi-order algorithm." At this point, the differences are minor, but we will see that the bracelet symbolism clarifies the situation when we generalize, in Section 3, from multiplication (or division) by 2 to multiplication (or division) by an arbitrary number t.

We will sometimes find it convenient to modify our way of looking at things by identifying each number a in $V(b)$ with $b - a$. This is more in the spirit of the geometric construction, where we considered only acute angles.

DEFINITION. Let $D(b) = \{a \in \mathbf{Z} | 0 < a < b/2 \text{ and } (a, b) = 1\}$. The function g_2: $D(b) \to D(b)$ is given by the rule

$$g_2(a) = \begin{cases} 2a & \text{if } 2a < b/2 \\ b - 2a & \text{if } 2a > b/2 \end{cases}.$$

To obtain a visual model of the structure induced by g_2, we need only draw a box around each pair $(a, b - a)$ in the graphs obtained above (see FIGURE 6 for the situation when $b = 17$).

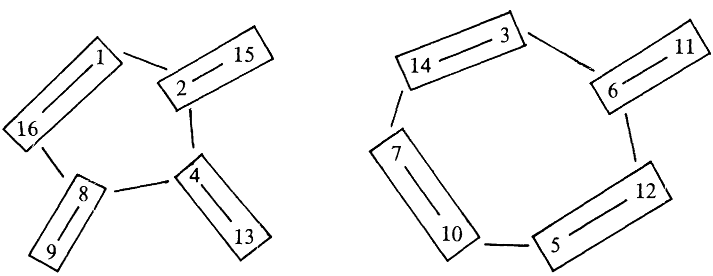


FIG. 6. Dominoes for $b = 17$.

We will call each such pair a *domino*, so that each bracelet can now be thought of simply as a cycle of dominoes. The cyclic geometric construction of Section 1 is represented by this graph: the domino containing $a \in D(b)$ corresponds to the acute angle $a\pi/b$. Note that as a consequence of this view, we obtain our first easy fact about bracelets.

PROPOSITION 1. *The weight of a bracelet is always even.*

If one computes (either by hand or, more easily, with a computer) the bracelet structures for some small values of b, especially for b prime, then one is led to an almost inexhaustible set of questions and conjectures. (To provide the reader with some examples for our discussion, we display a brief set of data in the Appendix.)

For example, do all the bracelets for a given b have the same weight? Do they all have sizes of the same parity? Is there any way to "compute" the number of bracelets, $B(2, b)$?

In the remainder of this article, then, we will explore such questions, with an emphasis on what can be said about the function B. In Section 3, we lay out the basic number-theoretic and group-theoretic way of looking at this problem, generalizing from 2 as the generator to an arbitrary t relatively prime to an arbitrary $b > 2$. In Section 4 we give a formula for computing the number of bracelets, their weights, and the parity of their sizes. Restricting ourselves to prime b in Section 5, we look more closely at how t and b determine the bracelet structure and specifically the number of bracelets. In Section 6 we extend the investigation to powers of primes and in Section 7 to arbitrary composite b; we will see that the set of prime factors of b tells essentially the whole story. Finally we close, in Section 8, with a few open-ended musings.

In addition to the work of Hilton and Pedersen, we mention that of J. B. Roberts [**13**], whose work anticipates some of ours, and H. P. Lawther, Jr. [**10**], who applied some related ideas to the splicing of telephone cables.

3. Some number theory and a slight generalization. In order to continue our investigation, it will be helpful to recall some elementary terminology and results from number theory and group theory.

Fix a (not necessarily odd) number $b > 2$. First note that we need not restrict ourselves to working with the set $V(b)$ of numbers relatively prime to b *and less than* b, because we are really working modulo b. Indeed, $V(b)$ is simply a collection of representatives for the set $\mathbf{Z}_b^*$ of reduced residue classes modulo b, which forms a finite abelian group under multiplication. Recall that the order of $\mathbf{Z}_b^*$, i.e., the cardinality of $V(b)$, is denoted by $\phi(b)$.

For t relatively prime to b, we consider the sequence $t, t^2, t^3, \ldots$. For some $n > 0$ we must have $t^n \equiv 1 \pmod{b}$; indeed, $t^{\phi(b)} \equiv 1 \pmod{b}$. The least such n is called the *order of* $t \bmod b$, and we shall denote it by $o(t, b)$. In fact, $t^n \equiv 1 \pmod{b}$ if and only if $o(t, b)$ is a divisor of n. Now it might happen that in the sequence $t, t^2, t^3, \ldots,$ the number $-1 \pmod{b}$ occurs before 1 does; this possibility has fundamental significance for the problems we are studying. Thus we make the following definition (which appears in [**6**] and, under different names, in [**10**] and [**13**], as well).

DEFINITION. Let t and b be relatively prime integers, $b > 2$. The *quasi-order* of t mod b, denoted $q(t, b)$, is the least positive integer k such that $t^k \equiv \pm 1 \pmod{b}$. If $t^{q(t,b)} \equiv -1 \pmod{b}$, then we call t *basic* $\pmod{b}$, and if $t^{q(t,b)} \equiv 1 \pmod{b}$, then we call t *nonbasic* $\pmod{b}$.

Note that $q(t, b)$ must equal either $o(t, b)$ or $o(t, b)/2$, since if $t^k \equiv \pm 1 \pmod{b}$, then $t^{2k} \equiv 1 \pmod{b}$. The following fact about quasi-order can be proved in a manner similar to the corresponding fact about order.

PROPOSITION 2. *Let t and b be relatively prime integers, $b > 2$. Then $t^n \equiv \pm 1 \pmod{b}$ if and only if $q(t, b)$ is a divisor of n.*

If $o(t, b) = \phi(b)$, then the set $\{t, t^2, t^3, \ldots, t^{\phi(b)}\}$ represents all the reduced residue classes mod b, and hence t generates $\mathbf{Z}_b^*$ under multiplication. In other words, $\mathbf{Z}_b^*$ is cyclic in this case. A well-known result of number theory, which we exploit in Section 5, states that $\mathbf{Z}_b^*$ is cyclic if and only if b equals 2, 4, p^n or $2p^n$ for p an odd prime and n a positive integer. In any case, the powers of $t \pmod{b}$ form a subgroup $\langle t \rangle$ of $\mathbf{Z}_b^*$.

In what follows we need to look at the subgroup $\langle t, -1 \rangle$ of $\mathbf{Z}_b^*$ generated by t and -1. If t is basic, then $-1 \in \langle t \rangle$, so in this case $\langle t, -1 \rangle = \langle t \rangle$ (hence our choice of the term *basic*). If t is nonbasic, then $\langle t, -1 \rangle$ contains all the elements of $\langle t \rangle$ together with their negatives. Thus in either case (but for different reasons), we obtain the following simple result.

PROPOSITION 3. *Let t and b be relatively prime integers, $b > 2$. Then $\{\pm t, \pm t^2, \pm t^3, \ldots, \pm t^{q(t,b)}\}$ is a set of representatives for $\langle t, -1 \rangle$, and the cardinality of $\langle t, -1 \rangle$ is $2q(t, b)$.*

Returning to the terminology of Section 2, we see that the bracelet containing 1 is exactly a collection of representatives for $\langle 2, -1 \rangle$. Furthermore, we can construct bracelets for values of t other than 2 simply by generalizing the definition of f_2 (and that of g_2), reducing all calculations modulo b. (Unfortunately, it no longer seems to be easy to characterize the charms.)

DEFINITION. Let t and b be relatively prime integers, $b > 2$. The function f_t: $V(b) \to V(b)$ and the function g_t: $D(b) \to D(b)$ are given by the following rules:

$$f_t(a) = \begin{cases} ta \bmod b & \text{if } a < b/2 \\ b - a & \text{if } a > b/2 \end{cases}$$

and

$$g_t(a) = \begin{cases} ta \bmod b & \text{if } ta \bmod b < b/2 \\ b - ta \bmod b & \text{if } ta \bmod b > b/2 \end{cases}.$$

As in the case $t = 2$, we let $B(t, b)$ be the number of bracelets formed in the construction. Note that Proposition 1 remains valid. As an example, we have the bracelet in FIGURE 7 for $b = 17$ and $t = 7$; thus $B(7,17) = 1$ and $q(7,17) = 8$.

Again, our function f_t (or g_t) and the bracelet graph replace the "quasi-order algorithm" and "symbol" of Hilton and Pedersen. Since all that is involved is multiplication modulo b, we avoid the cumbersome calculations and bookkeeping that they encounter when using the division-based approach when $t > 2$. On the other hand, their symbol does immediately determine the quasi-order and a criterion for whether t is basic (mod b).

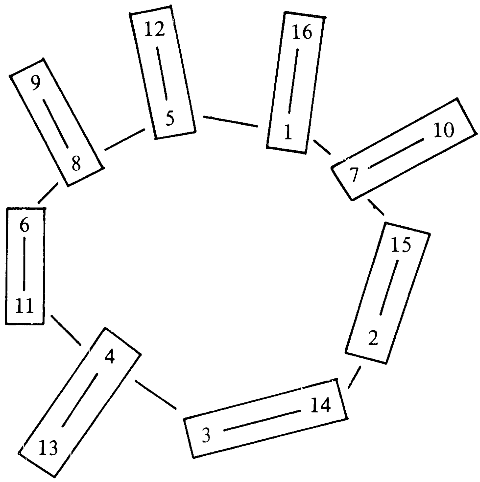


FIG. 7. Bracelet (with domino structure) for $b = 17$, $t = 7$.

4. A little group theory sheds some light. In Section 3 we saw that the bracelet containing 1 is just (a collection of representatives for) the subgroup $\langle t, -1\rangle$. In this section we apply some rudimentary group theory to discover what the remaining bracelets are, what the number of bracelets signifies, and how the domino structure can be interpreted. To avoid the awkward construction in parentheses in the first sentence of this paragraph, we will identify elements of $\mathbf{Z}_b^*$ with their representatives and think of the bracelets as actually containing elements of $\mathbf{Z}_b^*$.

THEOREM 1. *The bracelets for a given $b > 2$ and t relatively prime to b are precisely the cosets of $\langle t, -1\rangle$ in $\mathbf{Z}_b^*$. They all have weight $2q(t, b)$, and the number of bracelets is given by*

$$B(t, b) = \frac{\phi(b)}{2q(t, b)} = \textit{index of } \langle t, -1\rangle \text{ in } \mathbf{Z}_b^*.$$

Proof. For each $a \in \mathbf{Z}_b^*$, the bracelet containing a consists of all numbers (mod b) of the form at^i, $i = 1, 2, \ldots$. Thus it consists of precisely $\{\pm at, \pm at^2, \ldots, \pm at^{q(t,b)}\}$, which is the coset $a\langle t, -1\rangle$. The remaining statements follow from Proposition 3 and the definitions.

We next turn to an analysis of the sizes of the bracelets, i.e., the number of vertices in each bracelet which are not charms. We saw in FIGURE 5 that the sizes need not all be the same for a given b and t. On the other hand, the following theorem allows us to calculate their parity.

THEOREM 2. *The sizes of all the bracelets for a given $b > 2$ and t relatively prime to b have the same parity. This parity is even if t is basic $(\mathrm{mod}\, b)$ and $q(t, b)$ is odd, or if t is nonbasic $(\mathrm{mod}\, b)$ and $q(t, b)$ is even; and otherwise this parity is odd.*

Proof. Each step in a traversal of the cycle of a bracelet (i.e., the vertices which are not charms) corresponds either to multiplication by t (with a move to the next domino) or to multiplication by -1 (staying within the same domino). If s is the size of the bracelet, then after s steps (and not before), we will have returned to the starting point. If we performed k multiplications by t and $s-k$ multiplications by -1, then we must have (mod b)

$$at^k(-1)^{s-k} \equiv a$$

or simply

$$t^k \equiv (-1)^{s-k}.$$

Therefore, k must be $q(t, b)$ and $(-1)^{s-k}$ must be -1 or 1 according as whether t is basic or nonbasic (mod b), independent of the particular bracelet, i.e., independent of s. Thus the parity of $s-k$, and, hence, of s, depends only on b and t, in the manner stated. (It is also possible to give a proof based on a generalization of Gauss's Lemma due to Emma Lehmer [**11**].)

Since the size of a bracelet plus the number of its charms is even by Proposition 1, a statement similar to Theorem 2 holds as well for the number of charms.

Finally, we note that $D(b)$ can be viewed as a collection of representatives for the factor group $\mathbf{Z}_b^*/\langle -1\rangle$, which is essentially just the set of dominoes.

5. Some more number theory sheds some more light. In Section 4 we reduced questions about the sizes, weights, and numbers of bracelets to questions about $q(t, b)$ and about $t^{q(t,b)}$ mod b, i.e., whether t is basic or nonbasic (mod b). We now continue this study by classifying the bracelet structure into one of eight types, defined in terms of the sign of $t^{q(t,b)}$ mod b, the parity of $q(t, b)$, and the parity of $B(t, b)$. The following table shows the types, together with the smallest example in which $t = 2$ and b is prime (the dashes in the last column indicate that—as Theorem 3 will state—these types are impossible for prime b).

TABLE 1. TYPES OF BRACELET STRUCTURES.

Type 1.	t basic	$q(t, b)$ odd	$B(t, b)$ odd	$(b = 3)$
Type 2.	t nonbasic	$q(t, b)$ odd	$B(t, b)$ odd	$(b = 7)$
Type 3.	t basic	$q(t, b)$ even	$B(t, b)$ odd	$(b = 5)$
Type 4.	t nonbasic	$q(t, b)$ even	$B(t, b)$ odd	—
Type 5a.	t basic	$q(t, b)$ odd	$B(t, b)$ even	$(b = 281)$
Type 5b.	t nonbasic	$q(t, b)$ odd	$B(t, b)$ even	$(b = 73)$
Type 5c.	t basic	$q(t, b)$ even	$B(t, b)$ even	$(b = 17)$
Type 5d.	t nonbasic	$q(t, b)$ even	$B(t, b)$ even	—

By using quadratic residues and the fact that $\mathbf{Z}_b^*$ is cyclic for certain values of b (including all odd prime numbers), we will now determine, for such values of b, when each of these cases occurs. Recall that an integer x is a *quadratic* residue mod b if $x \equiv y^2$ (mod b) for some integer y.

THEOREM 3. *Suppose that* $\mathbf{Z}_b^*$ *is cyclic,* $b \neq 2, 4$ *(i.e.,* $b = p^n$ *or* $2p^n$, *for* p *an odd prime and* n *a positive integer), and suppose that* $(t, b) = 1$. *Then*

$$B(t, b) = \frac{p^{n-1}(p-1)}{2q(t, b)}.$$

Furthermore,

a) *if neither* t *nor* -1 *are quadratic residues* mod b, *then* $q(t, b)$ *is odd,* $t^{q(t,b)} \equiv -1 \pmod{b}$, $\langle t \rangle = \langle t, -1 \rangle \neq \langle -t \rangle$, *and* $B(t, b)$ *is odd (Type 1);*
b) *if* t *is a quadratic residue* mod b *but* -1 *is not, then* $q(t, b)$ *is odd,* $t^{q(t,b)} \equiv 1 \pmod{b}$, $\langle t \rangle \neq \langle t, -1 \rangle = \langle -t \rangle$, *and* $B(t, b)$ *is odd (Type 2);*
c) *if* -1 *is a quadratic residue* mod b *but* t *is not, then* $q(t, b)$ *is even,* $t^{q(t,b)} \equiv -1 \pmod{b}$, $\langle t \rangle = \langle t, -1 \rangle = \langle -t \rangle$, *and* $B(t, b)$ *is odd (Type 3);*
d) *if both* t *and* -1 *are quadratic residues* mod b, *then* $B(t, b)$ *is even (Type 5); if Type 5a occurs, then* $\langle t \rangle = \langle t, -1 \rangle \neq \langle -t \rangle$; *if Type 5b occurs, then* $\langle t \rangle \neq \langle t, -1 \rangle = \langle -t \rangle$; *and if Type 5c occurs, then* $\langle t \rangle = \langle t, -1 \rangle = \langle -t \rangle$;
e) *Type 4 and Type 5d cannot occur.*

Proof. Since $b = p^n$ or $2p^n$, it is easy to compute the order of $\mathbf{Z}_b^*$, namely $\phi(b) = p^{n-1}(p-1)$. Thus the displayed equation follows from Theorem 1.
a) Let the integer g be a generator of $\mathbf{Z}_b^*$, and write $t \equiv g^r \pmod{b}$. Since t is assumed not to be a quadratic residue mod b, r must be odd. Thus $t^{\phi(b)/2} \equiv (-1)^r \equiv -1 \pmod{b}$. We claim that, therefore, $q(t, b)$, which must be a divisor of $\phi(b)/2$ by Proposition 2, is odd. Indeed, since -1, which is congruent to $g^{\phi(b)/2}$, is assumed not to be a quadratic residue mod b, it must be that $\phi(b)/2$ is odd. It follows, then, that $t^{q(t,b)} \equiv -1 \pmod{b}$, i.e., t is basic. Combining all this we see that $B(t, b)$ is odd. Finally, since t is basic, it is clear that t generates $\langle t, -1 \rangle$; on the other hand, since $q(t, b)$ is odd, $(-t)^{q(t,b)} \equiv 1 \pmod{b}$, so $-t$ does not generate $\langle t, -1 \rangle$.
b) The proof is similar to part (a), except that we use $-t$ in place of t. This works because, given that t is a quadratic residue and -1 is not, $-t$ is not a quadratic residue.
c) The proof is similar to part (a), except that since $\phi(b)/2$ is even (by the assumption that -1 is a quadratic residue), $q(t, b)$ must absorb all the factors of 2 in $\phi(b)/2$ in order for $t^{\phi(b)/2}$ to be congruent to -1. Thus $q(t, b)$ is even, and, again, $B(t, b)$ is odd. Since t is basic, $\langle t \rangle = \langle t, -1 \rangle$; and since, in addition, $q(t, b)$ is even, $\langle -t \rangle = \langle t, -1 \rangle$, as well.
d) If $q(t, b)$ is odd, then since $\phi(b)/2$ is even, $B(t, b)$ is even, and the other statements follow as above. If $q(t, b)$ is even, then $t^{q(t,b)} \equiv -1 \pmod{b}$, since otherwise $t^{q(t,b)/2}$ would also be congruent to $\pm 1 \pmod{b}$, contradicting the definition of $q(t, b)$. (This last statement uses the fact that the number 1 has exactly two square roots in $\mathbf{Z}_b^*$ [**14**, p. 247].) Thus the order of t is $2q(t, b)$. It follows that $\langle t \rangle = \langle t, -1 \rangle = \langle -t \rangle$. Now since t is a quadratic residue, $t^{\phi(b)/2} \equiv 1 \pmod{b}$. Therefore $2q(t, b)$ divides $\phi(b)/2$, and so 2 divides $\phi(b)/(2q(t, b)) = B(t, b)$.
e) If $q(t, b)$ were even and $t^{q(t,b)} \equiv 1 \pmod{b}$, then $t^{q(t,b)/2} \equiv \pm 1 \pmod{b}$, as in

the proof of (d). This contradicts the definition of $q(t, b)$, so Type 4 and Type 5d cannot occur. (We will see later that they can occur when $\mathbf{Z}_b^*$ is not cyclic.)

We note the following corollary of Theorems 2 and 3.

COROLLARY 1. *Suppose that $\mathbf{Z}_b^*$ is cyclic, $b \neq 2, 4$ (i.e., $b = p^n$ or $2p^n$, for p an odd prime and n a positive integer), and suppose that $(t, b) = 1$. Then there are an even number of bracelets if and only if both t and -1 are quadratic residues $\bmod b$. Furthermore, the sizes of the bracelets are even if neither t nor -1 are quadratic residues, and odd if exactly one of t and -1 is a quadratic residue (the sizes may be either even or odd if both are quadratic residues).*

The determination of the quadratic residue status of 2 and $-1 \bmod b$ is always very simple in the cases we are considering. The following proposition is a standard exercise in number theory (see [**14**, pp. 254 and 256] for the flavor of the arguments involved).

PROPOSITION 4. *If $\mathbf{Z}_b^*$ is cyclic, $b \neq 2$ or 4, then -1 is a quadratic residue $\bmod b$ if and only if $b \equiv 1$ or $2 \pmod 4$. If $b = p^n$ for p an odd prime and n a positive integer, then 2 is a quadratic residue $\bmod b$ if and only if $p \equiv 1$ or $7 \pmod 8$.*

Combining Proposition 4 with Theorem 3, we see that, for $t = 2$ and b prime, types 1, 2, 3, and 5 occur when b is congruent to 3, 7, 5, and 1 (mod 8), respectively. Since by Dirichlet's Theorem [**14**, p. 375] there are an infinite number of primes in each of these congruence classes, we see that each of these types occurs infinitely often. In particular, *for $t = 2$ there are at least two bracelets for infinitely many values of b.* In case t is some number other than 2, results similar to Proposition 4 can be obtained using the Law of Quadratic Reciprocity, and again we will have an infinite number of occurrences of each of types 1, 2, 3, and 5. We leave it as an exercise to show that the residue class of b mod 12 determines the type when b is prime and $t = 3$.

Two obvious questions about the number of bracelets arise when one looks at the data in the Appendix: is there only one bracelet infinitely often, and can there be an arbitrarily large number of bracelets? In the remainder of this section, we will consider these questions for prime b.

The answer to both questions is an easy "yes" if we are willing to think of B as a function with two arguments, rather than thinking of t as a fixed parameter.

THEOREM 4. *For infinitely many pairs (t, b) with b prime, $B(t, b) = 1$. As a function of two variables, $B(t, b)$ is unbounded, even with b restricted to being prime.*

Proof. For the first claim, we need only observe that for prime b, we can choose the integer t to be a generator of the cyclic group $\mathbf{Z}_b^*$. For the second, we can make the number of bracelets arbitrarily large simply by taking t to be $b - 1$, since then the quasi-order of t will be 1, and so the number of bracelets will be $(b - 1)/2$ by Theorem 1.

The questions become much harder if we fix t (still insisting that b be prime). By Theorem 3, $B(2, b) = 1$ if and only if the integer 2 (or in some cases -2) is a generator of $\mathbf{Z}_b^*$. Number theorists call 2 (or -2) a *primitive root* in this case. It is conjectured, but not known, that 2 is a primitive root for infinitely many primes. If this conjecture were true (in a somewhat stronger form, since we would need to restrict ourselves to Type 1 or Type 3), then we could conclude that there is only one bracelet infinitely often. Similar statements could be made for other values of t.

Another long-standing conjecture is that there are an infinite number of primes p for which $b = 2p + 1$ is also prime [**15**, p. 129]. If p and b are such primes, then it is easy to show that $q(t, b) = p$, and therefore that $B(t, b) = 1$, for all t not congruent to $\pm 1 \pmod b$. In particular, if the conjecture is true, then for each $t > 1$ we would have $B(t, b) = 1$ infinitely often.

At the other end of the spectrum, $B(t, b)$ is large when $q(t, b)$ is small compared to b. The smallest $q(t, b)$ occurs when $t^{q(t,b)} = b \pm 1$, i.e., when $b = t^{q(t,b)} + 1$ or $b = t^{q(t,b)} - 1$. (There is currently a great deal of interest in the prime factorization of $t^k \pm 1$ for small t and large k; see [**1**].) In the particular case of $t = 2$, the questions of whether $2^k \pm 1$ are prime are well known. Prime numbers of the form $2^k + 1$ are the *Fermat primes*, and only five of them are known: 3, 5, 17, 257, and 65537. Thus, for example, $B(2,65537) = 65536/32 = 2048$. Prime numbers of the form $2^k - 1$ are the *Mersenne primes*, and only about 30 are known. (The largest Mersenne prime currently known is $2^{216091} - 1$.) Thus, for example, there are 315 bracelets when b is the Mersenne prime 8191. It is not known whether there are an infinite number of Fermat or Mersenne primes. If there are, then clearly $B(2, b)$ is unbounded for prime b. We can still derive this conclusion, however, by looking at the prime factors of the Fermat numbers.

THEOREM 5. *Let t be a fixed integer greater than* 1. *Then $B(t, b)$ is unbounded for prime numbers b.*

Proof. We use the following well-known lemma (whose proof is not hard—see [**15**, p. 343], for example): *If p is an odd prime divisor of $t^{2^n} + 1$, then $p = 2^{n+1}i + 1$ for some positive integer i.* Now for any positive integer M we can guarantee that $2^{n+1}i + 1$ is not prime for $1 \leqslant i \leqslant M$ by taking $n = \prod_{i=1}^{M} \phi(2i + 1)$. Indeed, since $2^{\phi(2i+1)} \equiv 1 \pmod{2i + 1}$, we have $2^n \equiv 1 \pmod{2i + 1}$, and, therefore, $2^{n+1}i + 1 \equiv 2i + 1 \equiv 0 \pmod{2i + 1}$. Thus we can take b to be an odd prime divisor of $t^{2^n} + 1$, which must exist since even if $t^{2^n} + 1$ is not odd, it is congruent to 2 (mod 4). By the lemma and the choice of n, we know that $b = 2^{n+1}i + 1$ for some $i > M$. On the other hand, the quasi-order of t is at most 2^n, since $t^{2^n} \equiv -1 \pmod b$, so by Theorem 3, $B(t, b)$ is at least $(b - 1)/2^{n+1} = i > M$.

6. Powers of primes. In the last section we looked at the quasi-order $q(t, b)$ and the number of bracelets $B(t, b)$, especially for odd prime b. We now make some further progress in the case in which b is a prime power. Of course Theorem 3 applies in the case of powers of odd primes. Here we want to relate $q(t, b)$ to $q(t, p)$ and $B(t, b)$ to $B(t, p)$ when $b = p^n$ where p is an odd prime, and also to compute $q(t, b)$ and $B(t, b)$ when $b = 2^n$.

We first state without proof a fundamental result relating the order of t mod b to the order of t mod pb [**12**, p. 364]. We need to assume at least that $t \neq \pm 1$ here, but for simplicity we will assume that $t > 1$.

PROPOSITION 5. *Let $b = p^n$ and $b' = p^{n+1}$, where p is an odd prime and $n \geqslant 1$. Assume $t > 1$ and $(t, b) = 1$. Then $o(t, b')$ is equal to either $o(t, b)$ or $po(t, b)$, and the second possibility holds precisely on the set of all $n \geqslant N(t, p)$ for some $N(t, p) \geqslant 1$ depending on t and p.*

In other words, as n increases, the order of t mod p^n eventually increases by the same factor as p^n increases, i.e., eventually $\phi(b)/o(t, b)$ becomes constant as n increases. In fact, it seems to be only rarely that $\phi(b)/o(t, b)$ increases at all, even as b goes from p to p^2. Primes p for which this does occur seem not to have been given a name in the number theory literature, so we will call them *Wieferich primes*, after an early twentieth-century mathematician who studied them in connection with Fermat's Last Theorem [**16**].

DEFINITION. A prime p is called *Wieferich* with respect to t if the order of t mod p^2 is the same as the order of t mod p.

The only Wieferich primes with respect to $t = 2$, less than 31 million, are 1093 and 3511 ([**8**] gives the complete—and very small—table of primes known (in 1965) to be Wieferich with respect to prime $t \leqslant 43$).

We now show that the behavior of the order of t mod p^n as n increases extends also to the behavior of the quasi-order, and hence that the number of bracelets for $b = p^n$ becomes constant for large enough n.

THEOREM 6. *Let $b = p^n$ and $b' = p^{n+1}$, where p is an odd prime and $n \geqslant 1$. Assume $t > 1$ and $(t, b) = 1$. Then*

$$\frac{q(t, b')}{q(t, b)} = \frac{o(t, b')}{o(t, b)} = \begin{cases} 1 & \text{if } n < N(t, p) \\ p & \text{if } n \geqslant N(t, p) \end{cases},$$

where $N(t, p)$ is the number guaranteed by Proposition 5.

Proof. The final equality follows from Proposition 5. As for the first, because of the remark above Proposition 2, there are two cases to consider.

i) Suppose $q(t, b) = o(t, b) = k$, so $t^k \equiv 1 \pmod b$. Hence k must be odd (otherwise $t^{k/2} \equiv \pm 1 \pmod b$, as in the proof of Theorem 3.d). By Proposition 5, $o(t, b')$ is also odd. Therefore, since $q(t, b')$ cannot be $o(t, b')/2$, it must equal $o(t, b')$, and the desired equality follows.
ii) Suppose $q(t, b) = o(t, b)/2 = k$. Thus $o(t, b) = 2k$, and so $o(t, b')$ is also even by Proposition 5. Therefore by the same reasoning as in (i), $q(t, b')$ must be $o(t, b')/2$ and not $o(t, b')$, and the desired equality follows.

COROLLARY 2. *Let p be an odd prime and $t > 1$ a number not divisible by p. If p is not Wieferich with respect to t, then*

$$B(t, p^n) = B(t, p).$$

In any case,

$$B(t, p^n) = p^{\min(n, N(t,p))-1} B(t, p),$$

where $N(t, p)$ is the number guaranteed by Proposition 5.

In other words, the number of bracelets for a prime power is eventually (and nearly always, it seems) independent of the power.

We now turn briefly to the case $b = 2^n$, with $n \geqslant 2$. Thus $\mathbf{Z}_b^*$ is not cyclic (unless $b = 4$), but it turns out [**14**, p. 205] that $\mathbf{Z}_b^*/\langle -1\rangle$ is. In contrast to Theorem 3, here usually neither t nor $-t$ generates the bracelet containing 1 (i.e., the subgroup $\langle t, -1\rangle$). We summarize what happens in this case but omit the proof. In terms of the classification given in TABLE 1, all types are possible here except Type 3 and Type 5c; almost always it is Type 4 or Type 5d.

THEOREM 7. *Let $b = 2^n$, $n \geqslant 2$, and let t be an odd integer. Then*

$$B(t, b) = \left(\frac{t \pm 1}{4}, \frac{b}{4}\right),$$

where the sign is chosen so that $(t \pm 1)/4$ is an integer. Also,

$$q(t, b) = \frac{2^{n-2}}{B(t, b)}.$$

Furthermore, t is nonbasic $\bmod b$ unless $t \equiv -1 \pmod b$.

7. The general composite case. We look finally at what can be said about $q(t, b)$ and $B(t, b)$ for b composite.

For simplicity, we will carry out the analysis in the case $b = p_1 p_2$, where p_1 and p_2 are distinct odd primes, but the argument will generalize to yield Theorems 8 and 9 below. Let $(t, b) = 1$, and let r_1 and r_2 be the quasi-orders of t mod p_1 and mod p_2, respectively. We will need to compare the highest powers of 2 dividing r_1 and r_2, so we write $r_1 = 2^{x_1} r_1'$ and $r_2 = 2^{x_2} r_2'$, where r_1' and r_2' are odd. Our goal is to determine $q(t, b)$ and $B(t, b)$ in terms of $q(t, p_i)$ and $B(t, p_i)$, $i = 1, 2$.

We first recall what is essentially the uniqueness part of the Chinese Remainder Theorem.

PROPOSITION 6. *If $a \equiv b \pmod{m_i}$ for $i = 1, 2, \ldots, n$, and if the m_i are pairwise relatively prime, then $a \equiv b \pmod{m_1 m_2 \cdots m_n}$.*

Since $t^{2r_i} \equiv 1 \pmod{p_i}$, clearly $t^{2[r_1, r_2]} \equiv 1 \pmod{p_i}$. Hence $t^{2[r_1, r_2]} \equiv 1 \pmod b$ by Proposition 6. Therefore, $q(t, b)$ is a divisor of $2[r_1, r_2]$. On the other hand, since $t^{q(t,b)} \equiv \pm 1 \pmod b$, we have $t^{q(t,b)} \equiv \pm 1 \pmod{p_i}$ for $i = 1, 2$. Thus by Proposition 2, $q(t, b)$ is a multiple of both r_1 and r_2 and hence of their least common multiple. Combining these two statements, we see that either $q(t, b) = [r_1, r_2]$ or

$q(t, b) = 2[r_1, r_2]$. To determine which formula applies, we need to look at the four cases.

Case 1. $t^{r_i} \equiv 1 \pmod{p_i}$ for $i = 1, 2$. Then $t^{[r_1, r_2]} \equiv 1 \pmod{b}$ by Proposition 6, so $q(t, b) = [r_1, r_2]$.

Case 2. $t^{r_i} \equiv -1 \pmod{p_i}$ for $i = 1, 2$. If r_1 and r_2 have the same highest power of 2 as a factor, i.e., $x_1 = x_2$ in the notation established above, then $[r_1, r_2]/r_i$ is odd, so $t^{[r_1, r_2]} \equiv -1 \pmod{p_i}$, for $i = 1, 2$. Thus $t^{[r_1, r_2]} \equiv -1 \pmod{b}$ by Proposition 6, and hence $q(t, b) = [r_1, r_2]$. If, however, $x_1 \neq x_2$, then one of $[r_1, r_2]/r_1$ and $[r_1, r_2]/r_2$ is even and the other is odd, so $t^{[r_1, r_2]}$ has different values reduced modulo p_1 and modulo p_2. Therefore $t^{[r_1, r_2]}$ cannot be congruent to either 1 or -1 (mod b). It follows that $q(t, b) = 2[r_1, r_2]$ in this subcase.

Case 3. $t^{r_1} \equiv -1 \pmod{p_1}$ and $t^{r_2} \equiv 1 \pmod{p_2}$. Using the same reasoning as in Case 2, we see that $q(t, b) = [r_1, r_2]$ if $x_1 < x_2$ and $q(t, b) = 2[r_1, r_2]$ if $x_1 \geqslant x_2$.

Case 4. $t^{r_1} \equiv 1 \pmod{p_1}$ and $t^{r_2} \equiv -1 \pmod{p_2}$. The situation is similar to Case 3.

As a corollary of this analysis, we see that, except for a factor of (r_1, r_2) and a possible factor of 2, the "number of bracelets" function is multiplicative. Indeed, since $B(t, b) = \phi(b)/(2q(t, b))$, we see that if $q(t, b) = [r_1, r_2]$, then

$$B(t, b) = \frac{\phi(b)}{2[r_1, r_2]} = \frac{(p_1 - 1)(p_2 - 1)}{2r_1r_2/(r_1, r_2)} = 2(r_1, r_2)B(t, p_1)B(t, p_2),$$

and if $q(t, b) = 2[r_1, r_2]$, then

$$B(t, b) = \frac{\phi(b)}{4[r_1, r_2]} = \frac{(p_1 - 1)(p_2 - 1)}{4r_1r_2/(r_1, r_2)} = (r_1, r_2)B(t, p_1)B(t, p_2).$$

As one example, let $t = 2$, $p_1 = 5$, and $p_2 = 7$. Then $r_1 = 2$, $r_2 = 3$, $t^{r_1} \equiv -1 \pmod{p_1}$, and $t^{r_2} \equiv 1 \pmod{p_2}$. Case 3 applies, with $x_1 > x_2$, so $q(2, 35) = 2[2, 3] = 12$, and $B(2, 35) = B(2, 5)B(2, 7) = 1$. For another example, suppose $t = 2$, $p_1 = 3$, and $p_2 = 11$. Then we are in Case 2 with $2^1 \equiv -1 \pmod 3$ and $2^5 \equiv -1 \pmod{11}$, so $q(2, 33) = [1, 5] = 5$ and hence $B(2, 33) = 2B(2, 3)B(2, 11) = 2$.

Nothing in the foregoing discussion except the calculations of $B(t, b)$ required that p_1 and p_2 were odd primes, only that each was greater than 2 and $(p_1, p_2) = 1$. Thus, generalizing in a straightforward way to arbitrary b, we obtain the following theorems, which give formulas for computing quasi-orders and numbers of bracelets for composite numbers in terms of these invariants for their prime-power factors. (In Corollary 3 we see that the number of bracelets ultimately does not even depend on the powers.) All eight types (see TABLE 1) are possible. We leave verification of the details to the reader.

THEOREM 8. *Let* $b = \prod_{i=1}^{n} p_i^{e_i}$, *where the* p_i *are distinct primes dividing* b *and* $b \not\equiv 2 \pmod 4$; *or let* $b = 2\prod_{i=1}^{n} p_i^{e_i}$, *where the* p_i *are distinct odd primes dividing* b.

Suppose $t > 1$ *and* $(t, b) = 1$. *Let* r_i *be the quasi-order of* $t \bmod p_i^{e_i}$, *and for each* i *write* $r_i = 2^{x_i} r_i'$, *where* r_i' *is odd. Then*

a) *if* t *is nonbasic* $(\bmod p_i^{e_i})$ *for all* i, *then* t *is nonbasic* $(\bmod b)$, *and* $q(t, b) = [r_1, r_2, \ldots, r_n]$;
b) *if* t *is basic* $(\bmod p_i^{e_i})$ *for all* i, *then*
 i) *if* $x_1 = x_2 = \cdots = x_n$, *then* t *is basic* $(\bmod b)$ *and* $q(t, b) = [r_1, r_2, \ldots, r_n]$, *and*
 ii) *otherwise,* t *is nonbasic* $(\bmod b)$ *and* $q(t, b) = 2[r_1, r_2, \ldots, r_n]$;
c) *if* t *is basic* $(\bmod p_i^{e_i})$ *for some* i, *and* t *is nonbasic* $(\bmod p_j^{e_j})$ *for some* j, *then* t *is nonbasic* $(\bmod b)$, *and*
 i) *if* $\max(x_1, x_2, \ldots, x_n) = x_i$ *for some* i *for which* t *is basic* $(\bmod p_i^{e_i})$, *then* $q(t, b) = 2[r_1, r_2, \ldots, r_n]$, *and*
 ii) *if* $\max(x_1, x_2, \ldots, x_n) > x_i$ *for every* i *for which* t *is basic* $(\bmod p_i^{e_i})$, *then* $q(t, b) = [r_1, r_2, \ldots, r_n]$.

THEOREM 9. *Let* $b = \prod_{i=1}^n p_i^{e_i}$, *where the* p_i *are distinct primes dividing* b *and* $b \not\equiv 2 \pmod 4$; *or let* $b = 2\prod_{i=1}^n p_i^{e_i}$, *where the* p_i *are distinct odd primes dividing* b. *Suppose* $t > 1$ *and* $(t, b) = 1$. *Let* r_i *be the quasi-order of* $t \bmod p_i^{e_i}$. *Then*

$$B(t, b) = 2^{n-1-\varepsilon} \frac{r_1 r_2 \cdots r_n}{[r_1, r_2, \ldots, r_n]} \prod_{i=1}^n B(t, p_i^{e_i}),$$

where ε *is either* 0 *or* 1, *depending on which of the cases in Theorem* 8 *applies*: $\varepsilon = 0$ *in cases* (a), $(b.i)$, *and* $(c.ii)$, *and* $\varepsilon = 1$ *in the cases* $(b.ii)$ *and* $(c.i)$.

Note that Theorem 9 also provides a simpler proof that $B(t, b)$ is unbounded as a function of (not necessarily prime) b, since we need only take n large to make $B(t,b)$ large.

Finally suppose $b = \prod_{i=1}^n p_i^{e_i}$, where the p_i are distinct primes dividing b, with $t > 1$ and $(t, b) = 1$. If the exponents e_i are large enough, then it is not hard to show by Theorems 5 and 6 and Corollary 2 that all the terms on the right-hand side of the displayed equation in Theorem 9 are independent of the exponents. Thus we have the following Corollary—which we find most remarkable—that *the number of bracelets for b is independent of the exponents in the prime factorization of b once the exponents get large enough.*

COROLLARY 3. *Let* $p_1, p_2, \ldots, p_n$ *be distinct primes, none of which divides* $t > 1$. *Then there exist constants* $E_1, E_2, \ldots, E_n$ *and* C *(depending on* $p_1, p_2, \ldots, p_n$ *and* t*) such that if* $b = \prod_{i=1}^n p_i^{e_i}$ *with* $e_i \geqslant E_i$ *for* $i = 1, 2, \ldots, n$, *then* $B(t, b) = C$.

As a comprehensive example, let us study $b = 720$, $t = 7$. Since 720 factors as $2^4 3^2 5$, we first apply Theorem 7 to obtain $B(7,16) = 2$, $q(7,16) = 2$, and $7^2 \equiv 1 \pmod{16}$. Then we apply Corollary 2 (since 3 is not Wieferich with respect to 7) to obtain $B(7,9) = B(7,3) = B(1,3) = 1$ and $q(7,9) = \phi(9)/2 = 3$; and we apply Theorem 3b (or calculate directly) to obtain $7^3 \equiv 1 \pmod 9$. Finally we compute that $B(7,5) = B(2,5) = 1$, $q(7,5) = q(2,5) = 2$, and $7^2 \equiv -1 \pmod 5$. If we combine all of these by Theorem 8.c.i, then we get $q(7,720) = 2[2,3,2] = 12$ and $7^{12} \equiv 1 \pmod{720}$; and applying Theorem 9, we see that $B(7,720) = 2^{3-1-1}(2 \cdot 3 \cdot 2/[2,3,2]) \cdot 2 \cdot 1 \cdot 1 = 8$. On the other hand, to see the force of Corollary 3, we have that if $e_1 \geqslant 5$, $e_2 \geqslant 1$, and $e_3 \geqslant 2$, then $B(7, 2^{e_1}3^{e_2}5^{e_3}) = 80$, since Theorem 8.c.ii applies and 5 is Wieferich with respect to 7 (N(in fact $7,5) = 2$).

Added in proof: We note that Theorem 8 answers a question raised by Man Keung Siu [**17**], who obtained for the case $t = 2$ a sharpening of a part of Theorem 3.

8. Afterword. We leave the reader with a comment, a generalization, and an open question.

In some sense our investigation (except for Section 7 and the end of Section 6) has focused on a structure which is extremely well understood: the cyclic group. Nevertheless, interesting questions emerged, and the progress was somewhat impeded by famous unsolved problems in elementary number theory, such as whether 2 is a primitive root for infinitely many primes. We found it fascinating that this problem touched so many corners of elementary number theory. It could serve as a fruitful area of "research" for an undergraduate following an introductory number theory course.

The quasi-order of t mod b was defined to be the least positive integer k such that t^k represents an element of the subgroup $\{-1,1\}$ of $\mathbf{Z}_b^*$. More generally, for other subgroups S of $\mathbf{Z}_b^*$, we could define the quasiorder of t mod b relative to S. Does anything interesting emerge?

Looking at the Appendix, we note that the sizes of the bracelets for a given b tend to be roughly equal, and that the bracelet containing 1 is usually the bracelet with the smallest size. Are any theorems lurking here?

Appendix: Some data. Listed here are the number of bracelets and the weights and sizes of the bracelets, for $t = 2$, b odd, $3 \leqslant b \leqslant 99$. The size of the bracelet containing 1 is marked with * when there is more than one bracelet.

b	$B(2,b)$	weights	sizes
3	1	2	2
5	1	4	3
7	1	6	5
9	1	6	4
11	1	10	8
13	1	12	9
15	1	8	6
17	2	8, 8	5*, 7
19	1	18	14
21	1	12	8
23	1	22	17
25	1	20	15
27	1	18	14
29	1	28	21
31	3	10, 10, 10	7*, 7, 9
33	2	10, 10	6*, 8

b	$B(2, b)$	weights	sizes
35	1	24	18
37	1	36	27
39	1	24	18
41	2	20, 20	13*, 17
43	3	14, 14, 14	10*, 10, 12
45	1	24	18
47	1	46	35
49	1	42	31
51	2	16, 16	12*, 12
53	1	52	39
55	1	40	30
57	2	18, 18	12*, 14
59	1	58	44
61	1	60	45
63	3	12, 12, 12	8*, 10, 10
65	4	12, 12, 12, 12	7*, 9, 9, 11
67	1	66	51
69	1	44	32
71	1	70	53
73	4	18, 18, 18, 18	11*, 13, 15, 15
75	1	40	30
77	1	60	44
79	1	78	55
81	1	54	40
83	1	82	62
85	4	16, 16, 16, 16	10*, 12, 12, 14
87	1	56	42
89	4	22, 22, 22, 22	15*, 15, 17, 19
91	3	24, 24, 24	16, 18*, 20
93	3	20, 20, 20	14*, 14, 16
95	1	72	54
97	2	48, 48	35*, 37
99	2	30, 30	22, 24*

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Integrals, an Introduction to Analytic Number Theory

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ILAN VARDI: I got my Ph.D. in Number Theory from M.I.T. in 1982, as a student of Dorian Goldfeld. I then spent a year at the Institute for Advanced Study. I was an acting assistant professor at Stanford from 1983 to 1985. After realizing that not everybody cared about Kloosterman Sums, I learned how to use a computer and tried out some applied math. I'm now interested in special functions related to determinants of Laplacians.

1. Introduction. An examination of Gradzhteyn and Ryzhyk's book of integral tables reveals a large number of difficult and obscure integral formulas. In my opinion one of the most remarkable is given on p. 532

$$\int_{\pi/4}^{\pi/2} \log\log\tan x\,dx = \frac{\pi}{2}\log\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right], \tag{1}$$

where

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt, \qquad s > 0$$

is the classical Γ-function. The reference given is to Bierens de Haan [**2**]. Failing to locate the proof of this formula, I decided to study equation (1) in some depth. It turns out that this formula requires some fairly involved analysis to prove, and also serves as a good example of how nontrivial number theory can be embedded in an integral formula.

The key to equation (1) is the *Dirichlet L-function*

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s}\cdots.$$

This is a well-known function; for example every calculus student knows the formula

$$L(1) = 1 - \frac{1}{3} + \frac{1}{5}\cdots = \frac{\pi}{4}.$$

Also, by the alternating series test $L(s)$ converges for $0 < s < 1$. However, much more is known and Hurwitz proved that $L(s)$ can be analytically continued to an entire function in the whole complex plane. He did this by proving the *functional equation*

$$L(1-s) = \left(\frac{2}{\pi}\right)^s \sin\frac{\pi s}{2}\Gamma(s)L(s). \tag{2}$$

What we will, in fact, show is that

$$\int_{\pi/4}^{\pi/2} \log\log\tan x\,dx = \frac{d}{ds}\Gamma(s)L(s)\Big|_{s=1} \tag{3}$$

Invoking the well-known formulas

$$\Gamma(1) = 1$$
$$\Gamma'(1) = -\gamma,$$

where γ is *Euler's constant*,

$$\gamma = \lim_{n\to\infty}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right\} = .5772156649015328606065 12\ldots,$$

equation (3) becomes

$$\int_{\pi/4}^{\pi/2} \log\log\tan x\,dx = -\gamma\frac{\pi}{4} + L'(1).$$

So the proof of equation (1) will consist of 2 parts: a) establishing (3) b) expressing $L'(1)$ in terms of logarithms of Γ-functions.

2. Proof of equation (3). We begin with a general *Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

which, if f is of polynomial growth, will converge absolutely in a half-plane $Re(s) > c$. We now use the technique first developed by Riemann to study the Riemann ζ-function

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt = \int_0^\infty e^{-nt}(nt)^{s-1}\,d(nt),$$

so

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nt}t^{s-1}\,dt.$$

Hence, by absolute convergence, one gets that for $Re(s) > c$

$$\Gamma(s)F(s) = \Gamma(s)\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \sum_{n=1}^{\infty}f(n)\int_0^\infty e^{-nt}t^{s-1}\,dt$$
$$= \int_0^\infty\left(\sum_{n=1}^{\infty}f(n)e^{-nt}\right)t^{s-1}\,dt.$$

Now let $z = e^{-t}$, this gives

$$\Gamma(s)F(s) = \int_0^1\left(\sum_{n=1}^{\infty}f(n)z^n\right)\left(\log\frac{1}{z}\right)^{s-1}\frac{dz}{z}.$$

Now we add the restriction that $f(n)$ be a *periodic* function. That is, there is a positive integer q such that $f(n+q) = f(n)$ for all n (for technical reasons also assume that $f(q) = 0$). With these assumptions we have that for $|z| < 1$

$$\sum_{n=1}^{\infty} f(n) z^n = \sum_{m=0}^{\infty} \sum_{n=1}^{q-1} f(mq+n) z^{mq+n}$$

$$= \frac{\sum_{n=1}^{q-1} f(n) z^n}{1 - z^q} = \frac{P(z,f)}{1-z^q},$$

where

$$P(z,f) = \sum_{n=1}^{q-1} f(n) z^n.$$

We have thus obtained the formula:

$$F(s)\Gamma(s) = \int_0^1 \frac{P(z,f)\left(\log \frac{1}{z}\right)^{s-1}}{1-z^q} \frac{dz}{z}. \tag{4}$$

This formula was first obtained by Dirichlet (see **[3]**) to derive his *class number formula* of which $L(1) = \pi/4$ is the simplest case. Differentiating equation (4) by Leibniz's rule gives

$$\frac{d}{ds} F(s)\Gamma(s) = \int_0^1 P(z,f) \frac{\left(\log \frac{1}{z}\right)^{s-1}}{1-z^q} \log\log\left(\frac{1}{z}\right) \frac{dz}{z}.$$

Now if $F(s)$ converges absolutely at $s = 1$ this will yield

$$F'(1) - \gamma F(1) = \int_0^1 P(z,f) \log\log\left(\frac{1}{z}\right) \frac{dz}{z}. \tag{5}$$

To prove equation (1) we let $q = 4$ and pick $f(n)$ to be the *quadratic character* (mod 4) that is

$$\chi_4(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 1 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \equiv 2 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

χ_4 is called the quadratic character (mod 4) because for $(n, 4) = 1$

$$\chi_4(n) = \begin{cases} 1 & \text{if } \exists x \text{ s.t. } x^2 \equiv n \pmod 4 \\ -1 & \text{otherwise,} \end{cases}$$

while $\chi_4(n) = 0$ if $(n, q) > 1$.

So we have

$$P(z, \chi) = z - z^3$$

and equation (5) becomes

$$L'(1) - \gamma\frac{\pi}{4} = \int_0^1 \frac{(z - z^3)\log\log\frac{1}{z}}{1 - z^4}\frac{dz}{z}$$

$$= \int_0^1 \log\log\left(\frac{1}{z}\right)\frac{dz}{1 + z^2} = \int_1^\infty \log\log u \frac{du}{1 + u^2}$$

$$= \int_{\pi/4}^{\pi/2} \log\log\tan x\, dx.$$

3. Evaluating $L'(1)$. It turns out that it is much easier, first, to evaluate $L'(0)$, then use the functional equation $L(s) \to L(1 - s)$ to obtain the value for $L'(1)$.

To compute $L'(0)$ we follow a method due to André Weil [7]. Let

$$\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)^s}, \qquad 0 < a \leq 1,$$

be the *Hurwitz ζ-function*. It is easily shown to converge for $Re(s) > 1$. Using the integral formula [**8**]

$$\Gamma(s)\zeta(a, s) = \int_0^\infty \frac{e^{-at}}{1 - e^t} t^{s-1}\, dt$$

one can show that $\zeta(a, s)$ can be analytically continued to the whole complex plane with only a simple pole at $s = 1$. The relevance of $\zeta(a, s)$ is due to the formula

$$L(s) = 4^{-s}\left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right];$$

thus evaluating $\zeta'(0, a)$ will yield the value of $L'(0)$ (for ease of notation we have written $\zeta'(s, a)$ to denote $\frac{\partial}{\partial s}\zeta(s, a)$). Weil's observation is the following: note that for $s > 1$

$$\zeta(s, a + 1) = \zeta(s, a) - \frac{1}{a^s},$$

thus

$$\zeta'(s, a + 1) = \zeta(s, a) + a^{-s}\log a,$$

and at $s = 0$

$$\zeta(0, a + 1) = \zeta(0, a) + \log a.$$

Letting

$$G(a) = e^{\zeta'(0, a)},$$

we see that $G(a)$ satisfies the functional equation

$$G(a+1) = aG(a).$$

Further, one has that

$$\frac{d^2}{da^2}\log G(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2} > 0, \quad \text{for } a > 0,$$

and that

$$G(a) \text{ is analytic for } a > 0.$$

These however are the exact conditions for the *Bohr-Mollerup Theorem* for the uniqueness of the Gamma function [**1**]. Thus one has that

$$G(a) = G(1)\Gamma(a).$$

One sees that $G(1) = \zeta'(0,1)$, and on noting that $\zeta(s,1) = \zeta(s)$, where $\zeta(s)$ is the Riemann ζ-function, one has

$$G(1) = \zeta'(0).$$

It is well known that $\zeta'(0) = -(1/2)\log 2\pi$ (e.g., [**6**], [**8**]), and so

$$\zeta'(0,a) = \log\frac{\Gamma(a)}{\sqrt{2\pi}}.$$

Substituting this in the formula for $L(s)$ one derives

$$L'(0) = \log\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - L(0)\log 4.$$

By the functional equation and $L(1) = \pi/4$ one gets that

$$L(0) = \frac{1}{2}.$$

And once again by the functional equation

$$\frac{2}{\pi}\frac{d}{ds}\Gamma(s)L(s)\bigg|_{s=1} = \frac{1}{2}\log 4 + \log\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{1}{2}\log\frac{\pi}{2},$$

and thus

$$L'(1) = \gamma\frac{\pi}{2} + \frac{\pi}{2}\log\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}.$$

This concludes the proof of equation (1).

4. More formulas! There are actually quite a number of identities in Gradzhteyn and Ryzhyk similar to (1). For example, there are

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\log\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}(2\pi)^{1/3}\right], \quad \text{page 571} \tag{6}$$

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\log 2\pi - \log\Gamma\left(\frac{1}{6}\right)\right], \quad \text{page 572.} \tag{7}$$

One sees that in equation (6) 3 plays the "key role" and in equation (7) 6 is the "magic number." To explain this one introduces *Dirichlet characters* (mod q)

$$\begin{aligned} &\chi \text{ is a Dirichlet character } (\text{mod } q) \text{ if} \\ \chi(1) &= 1 \\ \chi(n+q) &= \chi(n) && \forall n \\ \chi(n) &= 0 && \text{if } (n,q) > 1 \\ \chi(mn) &= \chi(m)\chi(n) && \forall m, n. \end{aligned}$$

The corresponding Dirichlet L-function is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad Re(s) > 0$$

and can be continued to an entire function if χ is not the *trivial character*

$$\chi_0(n) = 1 \quad \text{if } (n,q) = 1$$

Now the analogous character to χ_4 in equation (6) is the quadratic character (mod 3)

$$\chi_3(n) = \begin{cases} 0 & \text{if } n \equiv 0 \ (\text{mod } 3) \\ 1 & \text{if } n \equiv 1 \ (\text{mod } 3) \\ -1 & \text{if } n \equiv 2 \ (\text{mod } 3), \end{cases}$$

and in equation (7) the corresponding character is the quadratic character (mod 6) $\chi_6(n)$. Hence we have the L-functions

$$L(s,\chi_3) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} \cdots$$

$$L(s,\chi_6) = 1 - \frac{1}{5^s} + \frac{1}{7^s} \cdots.$$

The proofs of (6) and (7) are completely analogous to our proof of equation (1).

One can further explain how the numbers $3, 4, 6$ play the key roles in our formulas. First rewrite equation (1) in the same form as (6) and (7)

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \frac{\pi}{2}\log\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right].$$

Note that the solutions of x^2+1 are 4th roots of unity, i and $-i$, and one explains why $L(s,\chi_4)$ is involved by noting that it can be shown from the *Quadratic Reciprocity Theorem* that

$$\zeta_{\mathbf{Q}(i)}(s) = L(s,\chi_4)\zeta(s),$$

where $\zeta_{\mathbf{Q}(i)}(s)$ is the *Dedekind zeta function* of the field $\mathbf{Q}(i)$, and the classical definition (e.g., [**5**]) of the Dedekind ζ-function of the number field K is

$$\zeta_K(s) = \sum_{\substack{A\subseteq K \\ A \text{ ideal}}} \frac{1}{N(A)^s}.$$

Similarly, x^2+x+1 is the irreducible polynomial for the 3rd roots of unity, $-1/2 \pm \frac{\sqrt{-3}}{2}$, and, as above, $L(s,\chi_3)$ appears because

$$\zeta_{\mathbf{Q}(\sqrt{-3})}(s) = L(s,\chi_3)\zeta(s).$$

Similarly, x^2-x+1 gives the 6th roots of 1, so, as above, one expects $L(s,\chi_6)$ to play the central role.

5. Exercises.

1) Show that

$$\int_1^e \log(-\log\log y)\,dy = -\sum_{n=1}^{\infty}\frac{\log n}{n!} - \gamma e.$$

Hint: consider

$$L_f(s) = \sum_{n=1}^{\infty}\frac{1}{(n-1)!n^s}.$$

2) Find a similar formula for

$$\int_e^{e^e} \log(-\log\log\log y)\,dy.$$

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Dice with Fair Sums

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Consider the following situation. We are given two n-sided dice with the numbers 1 through n on their sides. These dice have the property that when both are rolled independently, the sum of the numbers showing behaves (as a random variable) as if the two dice were each fair. Can we conclude that each die is in fact fair? It surprised the authors to discover that the answer is sometimes no, depending on the particular value of n being considered. What follows is an exposition of this discovery.

Let X be a random variable. Suppose that X takes on only finitely many values and that each of these values is a positive integer. Let n be the largest integer such that $P(X = n) > 0$. We then call X an *n-sided die*. Define the polynomial $p_X(x) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1}$, where

$$r_k = \frac{P(X = k+1)}{P(X = n)} \quad \text{for } k = 0, 1, \ldots, n-1.$$

We call p_X the *companion polynomial* for the random variable X. It is easy to see that a polynomial is the companion polynomial for some die if and only if it is monic and has nonnegative coefficients. Then

$$P(X = k+1) = \frac{r_k}{r_0 + r_1 + \cdots + r_{n-1}} \quad \text{for } k = 0, 1, \ldots, n-1.$$

What we have called a companion polynomial is simply a normalized version of the generating function (see [**1**], Chapter 11). An n-sided die is *fair* if its companion is the *cyclic polynomial*

$$f_n(x) = 1 + x \cdots + x^{n-1}.$$

LEMMA 1.1. *Let X and Y be, respectively, n- and m-sided dice with companion polynomials p_X and p_Y. If X and Y are independent, then $W = X + Y$ is an $(n+m)$-sided die whose companion polynomial is $p_W(x) = xp_X(x)p_Y(x)$.*

Proof. We put

$$\begin{aligned}
p_X(x) &= r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} \\
p_Y(x) &= s_0 + s_1 x + \cdots + s_{m-1}x^{m-1} \\
p_X(x)p_Y(x) &= t_0 + t_1 x + \cdots + t_{n+m-2}x^{n+m-2} \\
p_W(x) &= u_0 + u_1 x + \cdots + u_{n+m-1}x^{n+m-1}.
\end{aligned}$$

Then for $k = 0, 1, \ldots, n+m-1$,

$$\begin{aligned}
u_k &= \frac{P(W = k+1)}{P(W = n+m)} = \sum_{i=1}^{n} \frac{P(X = i \text{ and } Y = k+1-i)}{P(W = n+m)} \\
&= \sum_{i=1}^{n} \frac{P(X = i)}{P(X = n)} \cdot \frac{P(Y = k+1-i)}{P(Y = m)} \\
&= \sum_{i=1}^{n} r_{i-1}s_{k-i} = \sum_{i=0}^{n-1} r_i s_{k-1-i}.
\end{aligned}$$

Note that because n and m are the greatest possible values of X and Y, respectively, it follows that

$$P(W = n+m) = P(X = n \text{ and } Y = m) = P(X = n)P(Y = m).$$

Above, we have examined all the ways in which $X + Y$ can equal $k+1$, and then used the independence of X and Y. Multiplication of the polynomials p_X and p_Y yields

$$t_{k-1} = \sum_{i=0}^{n-1} r_i S_{k-1-i},$$

where we interpret the $r_i = 0$ for $i > n$ or $i < 1$ and $s_i = 0$ for $i > m$ or $i < 1$. The equality $t_{k-1} = u_k$ implies that

$$p_W(x) = xp_X(x)p_Y(x).$$

Call a polynomial $p(x)$ *palindromic* if its coefficients are the same when read forwards and backwards. Equivalently, we may define a polynomial p of degree n to be palindromic if $p(x) = x^n p(x^{-1})$. It is not hard to see that the product of two palindromic polynomials is again palindromic.

If two random variables X and Y have the same distribution (or "law"), we write $L(X) = L(Y)$. Note that two dice have the same distribution if and only if they have the same companion polynomial.

The following result answers one half of our questions about dice whose sum is fair.

THEOREM 1.2. *Suppose that* $n \in \{1,2,\dots,9\} \cup \{11,13\}$. *Let* X *and* Y *be independent fair n-sided dice. Let* U *and* V *be independent dice, each of which has* $\leqslant n$ *sides. Suppose that* $L(U+V) = L(X+Y)$. *Then* U *and* V *are fair n-sided dice.*

Proof. This involves a certain amount of case-checking. For example, suppose $n = 6$. Let p_X, p_Y, p_U, p_V be the companion polynomials for X, Y, U, V, respectively. The companion polynomial for $X+Y$ and $U+V$ is (by lemma 1.1)

$$xp_U(x)p_V(x) = xp_X(x)p_Y(x) = x[f_6(x)]^2.$$

Now $f_6(x) = (1+x)q_1(x)q_2(x)$, where $q_1(x) = 1 + x + x^2$ and $q_2(x) = 1 - x + x^2$ are irreducible quadratic polynomials. Considering the degrees of these polynomials, we see that $deg(p_U) = deg(p_V) = 5$. Only two combinations of the irreducible factors $(1+x)$, $q_1(x)$, $q_2(x)$ are possible:

$$p_U(x) = p_V(x) = (1+x)q_1(x)q_2(x) = f_6(x)$$

or

$$p_U(x) = (1+x)(q_1(x))^2 \qquad p_V(x) = (1+x)(q_2(x))^2.$$

However, $(1+x)(q_2(x))^2$ has a couple of negative coefficients and hence cannot be the companion polynomial of any random variable. We are left with the case $p_U = p_V = p_X = p_Y$. Then U and V are fair 6-sided dice.

A similar procedure must be carried out for each n. One writes

$$f_n(x) = \begin{cases} (1+x)q_1(x)\cdots q_m(x) & n \text{ even} \\ q_1(x)\cdots q_m(x) & n \text{ odd} \end{cases},$$

where

$$m = \begin{cases} \dfrac{n-2}{2} & n \text{ even} \\[2ex] \dfrac{n-1}{2} & n \text{ odd} \end{cases},$$

and $q_1 \cdots q_m$ are irreducible quadratic factors. One has $q_p(x) = 1 - k_n(p)x + x^2$, where $k_n(p) = 2\cos(2p\pi/n)$ for $p = 1,\dots,m$. All of the real factorizations of $[f_n(x)]^2$ into products of these irreducible factors must be calculated.

For each n, these cases were examined (for the presence of negative coefficients) using a computer and MACSYMA, a large symbolic manipulation program developed at the M.I.T. Laboratory for Computer Science. In each case, it was determined that only the "fair" factorizations of $[f_n(x)]^2$ yielded polynomials with nonnegative coefficients. A copy of the computer output is available upon request.

The theorem is no longer true if we relax the condition that U and V have $\leqslant n$ sides. For example, suppose $n = 6$. Allowing the dice U and V to have more than six spots on a given side, we may consider a factorization

$$[f_6(x)]^2 = p_U(x)p_Y(x) = [(1+x)q_1(x)] \cdot \left[(1+x)q_1(x)(q_2(x))^2\right],$$
$$= [1 + 2x + 2x^2 + x^3][1 + x^2 + x^3 + x^4 + x^5 + x^7],$$

which yields the following.

Example 1.3. Let U and V be independent random variables with distributions given by

Event	$U = 1$	$U = 2$	$U = 3$	$U = 4$
Probability	1/6	2/6	2/6	1/6

Event	$V = 1$	$V = 2$	$V = 3$	$V = 4$	$V = 5$	$V = 6$	$V = 7$	$V = 8$
Probability	1/6	0	1/6	1/6	1/6	1/6	0	1/6

Then $W = U + V$ behaves as if it were the sum of two fair dice.

Event	$W = 2$	$W = 3$	$W = 4$	$W = 5$	$W = 6$	$W = 7$	$W = 8$	$W = 9$	$W = 10$	$W = 11$	$W = 12$
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

W. W. Funkenbusch pointed out to the authors that the dice U and V can be physically realized by taking a pair of ordinary dice and changing the number of spots on their sides.

U:	1	2	2	3	3	4

V:	1	3	4	5	6	8

Of course, according to our definition of "n-sided die," given above, the dice U and V here are "4-sided" and "8-sided," respectively.

THEOREM 1.4. *Let X, Y, U, V be n-sided dice. Suppose that X and Y are independent fair dice and that $L(U+V) = L(X+Y)$. Let p_U and p_V be the companion polynomials for U and V. Then*

1) *p_U and p_V are palindromic.*

2) *If $p_U = p_V$, then U and V are fair dice.*

3) *Suppose that the coefficients of p_U and p_V are all positive rational numbers. Then U and V are fair dice.*

Proof. 1) It follows from lemma 1.1 that $p_U(x)p_V(x) = [f_n(x)]^2$. Now the complex roots of f_n are the $n-1$ roots of unity, excluding $x = 1$. We pair each root with its complex conjugate and write

$$f_n(x) = \begin{cases} (x+1)q_1(x) \cdots q_m(x) & n \text{ even} \\ q_1(x) \cdots q_m(x) & n \text{ odd} \end{cases}$$

where $q_1, \ldots, q_m$ are irreducible quadratics of the form $q(x) = (x - \omega)(x - \omega) = 1 + kx + x^2$. The irreducible factors of p_U and p_V are thus all palindromic. So p_U and p_V are palindromic.

2) In case $p_U = p_V$, we see that $p_U(x)p_V(x) = [p_U(x)]^2 = [f_n(x)]^2$, so that $(p_U - f_n)(p_U + f_n)$ is the zero polynomial. It follows that $p_U = p_V = f_n$, so that U and V are fair dice.

3) The roots of $p_U(x)$ and $p_V(x)$ are algebraic integers. Gauss' lemma [**3**; Theorem 9.7] implies that the coefficients of p_U and p_V are ordinary ("rational") integers. Put

$$p_U(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-2}x^{n-2} + r_{n-1}x^{n-1}$$

$$p_V(x) = s_0 + s_1x + s_2x^2 + \cdots + s_{n-2}x^{n-2} + s_{n-1}x^{n-1}$$

$$[f_n(x)]^2 = t_0 + t_1x + t_2x^2 + \cdots + t_{2n-3}x^{2n-3} + t_{2n-2}x^{2n-2}.$$

Then $t_k = \min\{k+1, 2n-1-k\} = r_0s_k + r_1s_{k-1} + \cdots + r_ks_0$. Now p_U and p_V are monic, and from part 1, palindromic. So $r_{n-1} = 1 = r_0$ and $s_{n-1} = 1 = s_0$. Then $t_1 = 2 = r_0s_1 + r_1s_0 = s_1 + r_1$. Since all the coefficients are assumed to be positive, the only possibility is that $1 = s_1 = s_{n-2}$ and $1 = r_1 = r_{n-2}$. Then consider $t_2 = 3 = r_0s_2 + r_1s_1 + r_2s_0 = s_2 + 1 + r_2$, forcing $s_2 = s_{n-3} = 1$ and $r_2 = r_{n-3} = 1$. One proceeds inductively to obtain $r_0 = r_1 = \cdots = r_{n-1} = s_0 = s_1 = \cdots = s_{n-1} = 1$. Both U and V are fair dice.

In part 3 of theorem 1.4, the assumption of rationality for the coefficients cannot be eliminated. In theorem 1.14 below, we exhibit independent, but unfair, n-sided dice whose sum behaves as if the dice were fair.

One further note: the coefficients $r_0, r_1, \ldots, r_{n-1}$ are all rational if and only if $s_0, s_1, \ldots, s_{n-1}$ are all rational. This is because, when $r_0, r_1, \ldots, r_{n-1}$ are known, the coefficients $s_0, s_1, \ldots, s_{n-1}$ may be obtained by solving the system of linear equations

$$r_0s_k + r_1s_{k-1} + \cdots + r_ks_0 = \min\{k+1, 2n-1-k\}$$

using Cramer's Rule. Since the matrix of coefficients and the $\min\{k+1, 2n-1-k\}$ are rational, it follows that $s_0, \ldots, s_{n-1}$ are rational.

We now turn our attention to Chebyshev polynomials. They are a most useful computational tool for handling the factorization of cyclic polynomials.

The Chebyshev polynomials $S_0, S_1, S_2, \ldots$ (actually, these are versions of what are termed "Chebyshev polynomials of the first kind") have many uses in mathematical analysis, and there are many equivalent methods of defining them (see, e.g. [**2**] or [**4**]). One way is to set $S_0(x) = 1$ and $S_1(x) = x$ and use the recurrence relation $S_{j+1}(x) = xS_j(x) - S_{j-1}(x)$. A matrix formulation is

$$\begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix}^n = \begin{bmatrix} -S_{n-2}(x) & -S_{n-1}(x) \\ S_{n-1}(x) & S_n(x) \end{bmatrix}.$$

We summarize the basic facts about Chebyshev polynomials that we shall need in

LEMMA 1.5. *Let $S_0, S_1, S_2, \ldots$ be the sequence of Chebyshev polynomials.*
1) $(x-2)\sum_{m=0}^{j-1} S_m(x) = S_j(x) - S_{j-1}(x) - 1$ *for* $j = 1, 2, \ldots$.
2) $(\sin\theta)S_{j-1}(2\cos\theta) = \sin(j\theta)$ *for* $j = 1, 2, \ldots$.

3) *Let* α *and* β *be real numbers in* $[-1,1]$ *such that neither* $1-\alpha z+z^2$ *nor* $1-\beta z+z^2$ *vanishes for complex values* z *in the open disk* $|z|<1/2$. *Then for* $|z|<1/2$

$$\frac{1-\alpha z+z^2}{1-\beta z+z^2}=1+(\beta-\alpha)\sum_{j=0}^{\infty}S_j(\beta)z^{j+1}.$$

Proof. 1) Define the matrix

$$A=\begin{bmatrix}0 & -1\\ 1 & x\end{bmatrix},$$

and consider the identity

$$\begin{aligned}I+A+A^2+\cdots+A^{n-1}&=(A-I)^{-1}(A^n-I)\\&=\frac{1}{x-2}\begin{bmatrix}1-x & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-S_{n-2}(x)-1 & -S_{n-1}(x)\\ S_{n-1}(x) & S_n(x)-1\end{bmatrix}.\end{aligned}$$

Inspection of the lower right-hand entry yields the desired result.

2) This well-known identity follows from the basic recurrence relation by a simple induction argument.

3) Define a function of two variables

$$g(z,\beta)=\sum_{j=0}^{\infty}S_j(\beta)z^j.$$

Then

$$\begin{aligned}(\beta-z)g(z,\beta)&=\beta+\sum_{j=1}^{\infty}\left(\beta S_j(\beta)-S_{j-1}(\beta)\right)z^j\\&=\beta+\sum_{j=1}^{\infty}S_{j+1}(\beta)z^j\end{aligned}$$

by the recurrence relation. So

$$z(\beta-z)g(z,\beta)-\beta z=\sum_{j=1}^{\infty}S_{j+1}(\beta)z^{j+1}=g(z,\beta)-1-\beta z,$$

which leads to

$$g(z,\beta)=\sum_{j=0}^{\infty}S_j(\beta)z^j=\frac{1}{z^2-\beta z+1}.$$

Now

$$\frac{1-\alpha z+z^2}{1-\beta z+z^2}-1=\frac{(\beta-\alpha)z}{1-\beta z+z^2}=(\beta-\alpha)zg(z,\beta),$$

so that

$$\frac{1-\alpha z+z^2}{1-\beta z+z^2}=1+(\beta-\alpha)\sum_{j=0}^{\infty}S_j(\beta)z^{j+1}$$

as desired. It remains only to note that $|S_n(\beta)|\leqslant 2^n$ whenever $|\beta|\leqslant 1$ (proof by induction), so that the power series we have been manipulating so freely in fact converges for $|z|<1/2$. (The reader might want to press on and show convergence for $|z|<1$, which does hold.)

In view of theorem 1.2, it involves no loss of generality in considering n-sided dice with fair sums where $n\geqslant 10$. With this restriction on n, we introduce some notation. Given $n\geqslant 10$, choose $\{p,q\}\subseteq\{1,2,\ldots,m\}$, where $p\neq q$ and

$$m=\begin{cases}\dfrac{n-1}{2} & \text{if } n \text{ is odd}\\[2mm] \dfrac{n-2}{2} & \text{if } n \text{ is even.}\end{cases}$$

Define $k_n(p)=2\cos(2\pi p/n)$ and set

$$h_n(p,q;x)=\frac{1-k_n(p)x+x^2}{1-k_n(q)x+x^2}f_n(x).$$

Put

$$\begin{aligned}\lambda_n(p,q)&=\cos(2\pi q/n)-\cos(2\pi p/n)\\&=2\sin(\pi(p+q)/n)\sin(\pi(p-q)/n),\end{aligned}$$

and define $c_j=c_j(p,q,n)$ by

$$h_n(p,q;x)=c_0+c_1x+c_2x^2+\cdots+c_{n-1}x^{n-1}$$

and $d_j=d_j(p,q,n)=c_j(q,p,n)$, so that

$$h_n(q,p;x)=d_0+d_1x+d_2x^2+\cdots+d_{n-1}x^{n-1}.$$

Define

$$\phi_j(p)=\sin\left(\frac{2p\pi}{n}\right)+\sin\left(\frac{2jp\pi}{n}\right)-\sin\left(\frac{2(j+1)p\pi}{n}\right)$$

THEOREM 1.6. *With notation as above, the following identities hold*:

$$c_j=1+\lambda_n(p,q)\left(1-\cos\left(\frac{2q\pi}{n}\right)\right)^{-1}\phi_j(q)\left(\sin\left(\frac{2q\pi}{n}\right)\right)^{-l}$$

$$d_j=1+\lambda_n(q,p)\left(1-\cos\left(\frac{2p\pi}{n}\right)\right)^{-1}\phi_j(p)\left(\sin\left(\frac{2p\pi}{n}\right)\right)^{-1}$$

Proof. Since $1 - k_n(p)z + z^2$ and $1 - k_n(q)z + z^2$ are nonzero in the complex disk $|z| < 1/2$, we have

$$\frac{1 - \alpha x + x^2}{1 - \beta x + x^2} = 1 + (\beta - \alpha) \sum_{j=0}^{\infty} S_j(\beta) x^{j+1} \qquad |x| < 1/2$$

for $\alpha = k_n(p)$ and $\beta = k_n(q)$ or *vice versa* (lemma 1.5.3). So

$$\begin{aligned} h_n(p, q; x) &= \{1 + 2\lambda_n(p, q)(S_0(k_n(q))x + S_1(k_n(q))x^2 + \cdots)\} \\ &\quad \times \{1 + (x + x^2 + \cdots + x^{n-1})\} \\ &= \{1 + A(x)\} \times \{1 + B(x)\} \\ &= \{1 + A(x) + B(x) + A(x)B(x)\}. \end{aligned}$$

We calculate

$$\begin{aligned} A(x)B(x) &= 2\lambda_n(p, q)(S_0(k_n(q))x + S_1(k_n(q))x^2 + \cdots) \\ &\quad \times (x + x^2 + \cdots + x^{n-1}) \\ &= \lambda_n(p, q)(S_0(k_n(q))x^2 + (S_0(k_n(q)) + S_1(k_n(q)))x^3 + \cdots). \end{aligned}$$

Then

$$\begin{aligned} h_n(p, q; x) = 1 &+ x(2\lambda_n(p, q)S_0(k_n(q)) + 1) \\ &+ x^2(2\lambda_n(p, q)(S_0(k_n(q)) + S_1(k_n(q))) + 1) \\ &+ x^3(2\lambda_n(p, q)(S_0(k_n(q)) + S_1(k_n(q)) + S_2(k_n(q))) + 1) \\ &+ \cdots, \end{aligned}$$

so that

$$\begin{aligned} c_j &= 1 + 2\lambda_n(p, q) \sum_{m=0}^{j-1} S_m(k_n(q)) \\ &= 1 + 2\lambda_n(p, q)(k_n(q) - 2)^{-1}(S_j(k_n(q)) - S_{j-1}(k_n(q)) - 1) \end{aligned}$$

for $j = 0, 1, \ldots, n - 1$ (lemma 1.5.1). Note that this infinite series expansion of $h_n(p, q; x)$ is in fact just a polynomial. Application of lemma 1.5.2 shows that

$$c_j = 1 + 2\lambda_n(p, q)(k_n(q) - 2)^{-1}\left(\frac{\sin((j + 1)2\pi q/n)}{\sin(2\pi q/n)} - \frac{\sin(2\pi qj/n)}{\sin(2\pi q/n)} - 1\right)$$

$$= 1 + \lambda_n(p, q)\left(1 - \cos\left(\frac{2\pi q}{n}\right)\right)^{-1} \phi_j(q)(\sin(2\pi q/n))^{-1}$$

as desired. The formula for d_j is obtained by reversing the roles of p and q.

THEOREM 1.7. *With notation as above, one has $c_j > 0$ if and only if*

$$\lambda_n(q, p)\left(\cos\left(\frac{q\pi}{n}\right) - \cos\left(\frac{(2j + 1)q\pi}{n}\right)\right) < \sin\left(\frac{q\pi}{n}\right)\sin\left(\frac{2q\pi}{n}\right)$$

and $d_j > 0$ if and only if

$$\lambda_n(q, p)\left(\cos\left(\frac{(2j+1)p\pi}{n}\right) - \cos\left(\frac{p\pi}{n}\right)\right) < \sin\left(\frac{p\pi}{n}\right)\sin\left(\frac{2p\pi}{n}\right).$$

Proof. Using theorem 1.6, we note the following equivalent statements, each of which is equivalent with $c_j > 0$:

$$-\lambda_n(p, q)\phi_j(q) < \left(1 - \cos\left(\frac{2\pi q}{n}\right)\right)\sin\left(\frac{2\pi q}{n}\right)$$

$$-\lambda_n(p, q)\phi_j(q) < 2\left(\sin\left(\frac{q\pi}{n}\right)\right)^2 \sin\left(\frac{2\pi q}{n}\right)$$

$$-\lambda_n(p, q)\left(2\sin\left(\frac{q\pi}{n}\right)\cos\left(\frac{q\pi}{n}\right) + 2\cos\left(\frac{(2j+1)q\pi}{n}\right)\sin\left(-\frac{q\pi}{n}\right)\right)$$

$$< 2\left(\sin\left(\frac{q\pi}{n}\right)\right)^2 \sin\left(\frac{2\pi q}{n}\right)$$

$$\lambda_n(q, p)\left(\cos\left(\frac{q\pi}{n}\right) - \cos\left(\frac{(2j+1)q\pi}{n}\right)\right) < \sin\left(\frac{q\pi}{n}\right)\sin\left(\frac{2q\pi}{n}\right),$$

as desired. The result for d_j follows in like manner.

COROLLARY 1.8. *With notation as above, suppose that $p < q$, so that $\lambda_n(p, q) < 0 < \lambda_n(q, p)$. Then $h_n(p, q; x)$ has strictly positive coefficients if*

$$2\sin\left(\frac{(p+q)\pi}{n}\right)\sin\left(\frac{(q-p)\pi}{n}\right) < \tan\left(\frac{q\pi}{2n}\right)\sin\left(\frac{2q\pi}{n}\right), \tag{*}$$

and $h_n(q, p; x)$ has strictly positive coefficients if

$$2\sin\left(\frac{(p+q)\pi}{n}\right)\sin\left(\frac{(q-p)\pi}{n}\right)\tan\left(\frac{p\pi}{2n}\right) < \sin\left(\frac{2p\pi}{n}\right). \tag{**}$$

Proof. Assuming (*) holds, we have

$$\lambda_n(q, p)\left(\cos\left(\frac{q\pi}{n}\right) - \cos\left(\frac{(2j+1)q\pi}{n}\right)\right) < \lambda_n(q, p)\left(\cos\left(\frac{q\pi}{n}\right) + 1\right)$$

$$< \sin\left(\frac{2q\pi}{n}\right)\tan\left(\frac{q\pi}{2n}\right)\left(\cos\left(\frac{q\pi}{n}\right) + 1\right)$$

$$= \sin\left(\frac{2q\pi}{n}\right)\sin\left(\frac{q\pi}{n}\right),$$

so that, by theorem 1.7, each c_j is positive. Now assume (**) and note

$$\begin{aligned}\lambda_n(q,p)\left(\cos\left(\frac{(2j+1)p\pi}{n}\right)-\cos\left(\frac{p\pi}{n}\right)\right) &< \lambda_n(q,p)\left(1-\cos\left(\frac{p\pi}{n}\right)\right)\\ &< \lambda_n(q,p)\tan\left(\frac{p\pi}{2n}\right)\sin\left(\frac{p\pi}{n}\right)\\ &< \sin\left(\frac{2p\pi}{n}\right)\sin\left(\frac{p\pi}{n}\right),\end{aligned}$$

so that, by theorem 1.7, each d_j is positive.

Define a function Q by the rule

$$Q(a,b,n)=\frac{\sin(a\pi/n)}{2\sin(b\pi/n)}.$$

COROLLARY 1.9. *With notation as above, assume that $q=p+1<n/2$. Then $h_n(p,q;z)$ has strictly positive coefficients if*

$$\sin\left(\frac{\pi}{n}\right)<\tan\left(\frac{q\pi}{2n}\right)Q(2p+2,2p+1,n),$$

and $h_n(q,p;x)$ has strictly positive coefficients if

$$\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{p\pi}{2n}\right)<Q(2p,2p+1,n).$$

Proof. Immediate consequence of corollary 1.8.

COROLLARY 1.10. *Given $n\geqslant 10$, take $p=(n-3)/2$, $q=(n-1)/2$ if n is odd; take $p=(n-4)/2$, $q=(n-2)/2$ if n is even. Define $\delta(n)\in\{1,2\}$ by $2p+\delta(n)=n-2$. Then $h_n(p,q;x)$ has strictly positive coefficients if*

$$\sin\left(\frac{\pi}{n}\right)<\tan\left(\frac{q\pi}{n}\right)Q(\delta(n),\delta(n)+1,n).$$

Also $h_n(q,p;x)$ has strictly positive coefficients if

$$\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{p\pi}{2n}\right)<Q(\delta(n)+2,\delta(n)+1,n).$$

Proof. This follows from corollary 1.9, noting that

$$\sin\left(\frac{(2p+2)\pi}{n}\right)=\sin\left(\frac{\delta(n)\pi}{n}\right)$$

$$\sin\left(\frac{(2p+1)\pi}{n}\right)=\sin\left(\frac{(\delta(n)+1)\pi}{n}\right)$$

$$\sin\left(\frac{2p\pi}{n}\right)=\sin\left(\frac{(\delta(n)+2)\pi}{n}\right).$$

THEOREM 1.11. *Given $n \geqslant 10$, put $\delta(n) = 1$ if n is odd and $\delta(n) = 2$ if n is even. Set $m = (n - \delta(n))/2$. Then $h_n(m, m-1; x)$ has strictly positive coefficients.*

Proof. By corollary 1.10, it suffices to show that

$$\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{p\pi}{2n}\right) < Q(\delta(n)+2, \delta(n)+1, n).$$

Now $Q(\delta(n)+2, \delta(n)+1, n) > 1/2$ because

$$\sin\left(\frac{(\delta(n)+2)\pi}{n}\right) > \sin\left(\frac{(\delta(n)+1)\pi}{n}\right) > 0.$$

Also,

$$\tan\left(\frac{p\pi}{2n}\right) < \tan\left(\frac{\pi}{4}\right) = 1$$

because $2p \leqslant n - 3$. Thus

$$\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{p\pi}{2n}\right) < \sin\left(\frac{\pi}{n}\right) < 1/2 < Q(\delta(n)+2, \delta(n)+1, n),$$

as desired.

LEMMA 1.12. *If $n \geqslant 10$, then $Q(2, 3, n) > 1/3$.*

Proof. For $\theta = \pi/n$, we need to show that $(\sin 2\theta)/(\sin 3\theta) > 2/3$. Now

$$\cos\theta + \sin\theta\left(\frac{\cos 2\theta}{\sin 2\theta}\right) < 3/2$$

because $(\cos 2\theta)/\cos\theta < 1$, and hence

$$\sin 3\theta = \sin 2\theta\cos\theta + \sin\theta\cos 2\theta < (3/2)\sin 2\theta.$$

The results follow, noting that $0 < \sin 2\theta < \sin 3\theta$.

THEOREM 1.13. *Given $n \geqslant 10$, put $\delta(n) = 1$ if n is odd and $\delta(n) = 2$ if n is even. Set $m = (n - \delta(n))/2$. Then $h_n(m-1, m; x)$ has strictly positive coefficients for $n \geqslant 14$ and for $n = 12$.*

Proof. By corollary 1.9, it suffices to show that

$$\sin\left(\frac{\pi}{n}\right) < \tan\left(\frac{q\pi}{2n}\right)Q(\delta(n), \delta(n)+1, n),$$

where $q = m$. For n odd, this is the same as $\sin(2\pi/n) < (1/2)\tan(m\pi/2n)$. For n even, lemma 1.12 serves to establish the sufficiency of $\sin(\pi/n) < (1/3)\tan(\mathrm{m}\pi/2n)$. In either case, success at a given value n_0 implies success at all $n \geqslant n_0$ such that $n \equiv n_0 \pmod 2$. Thus the proof is complete for $n \geqslant 14$ upon inspection of the following table of calculator approximations:

n	$\sin(\pi/n)$	$\sin(2\pi/n)$	$(1/3)\tan(m\pi/2n)$	$(1/2)\tan(m\pi/2n)$
12	.2588		.2558	
13		.4647		.4430
14	.2225		.2658	
15		.4067		.4502

For $n = 12$, we note

$$\sin\left(\frac{\pi}{12}\right) < .26 < .27 < \tan\left(\frac{5\pi}{24}\right)Q(2,3,12).$$

Finally, we are ready for a result complementary to theorem 1.2. Combined, the two results provide a complete answer to the question of whether unfair n-sided dice can have a fair sum.

THEOREM 1.14. *Suppose that $n \in \{10, 12\} \cup \{14, 15, \dots\}$. Let X and Y be independent fair n-sided dice. There are independent n-sided dice U and V such that $L(U+V) = L(X+Y)$, but U and V are not fair.*

Proof. **Case 1**: Suppose $n \in \{12\} \cup \{14, 15, \dots\}$. Put $\delta = 1$ if n is odd and $\delta = 2$ if n is even. Set $m = (n-\delta)/2$. By theorems 1.11 and 1.13, $h_n(m-1, m; x)$ and $h_n(m, m-1; x)$ are polynomials with strictly positive coefficients. So let U and V be n-sided dice whose companion polynomials are

$$p_U(x) = h_n(m-1, m; x) \qquad p_V(x) = h_n(m, m-1; x).$$

Then $p_U(x)p_V(x) = [f_n(x)]^2$, so that $L(U+V) = L(X+Y)$ as desired.

Case 2: Suppose $n = 10$. Computation of

$$\begin{bmatrix} -S_{r-2}(x) & -S_{r-1} \\ S_{r-1}(x) & S_r(x) \end{bmatrix}^2 = \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix}^{2r} = \begin{bmatrix} -S_{2r-2}(x) & -S_{2r-1}(x) \\ S_{2r-1} & S_{2r}(x) \end{bmatrix}$$

yields the identity $S_{2r} = (S_r - S_{r-1})(S_r + S_{r-1})$. Taking $r = 2$ gives

$$S_4(2\cos\theta) = 0 \quad \text{iff} \begin{cases} (2\cos\theta)^2 + 2\cos\theta - 1 = 0 \\ \quad\text{or} \\ (2\cos\theta)^2 - 2\cos\theta - 1 = 0 \end{cases}$$

Thus we may tabulate the values of $k_{10}(p) = 2\cos(2\pi p/10)$:

$$k_{10}(1) = (1+\sqrt{5})/2 \qquad k_{10}(2) = (-1+\sqrt{5})/2$$

$$k_{10}(3) = (1-\sqrt{5})/2 \qquad k_{10}(4) = (-1-\sqrt{5})/2.$$

(This explicit form for $\cos(36°)$ does not seem to be widely known.) A manageable hand computation yields

$$h_{10}(3,4;x) = (1+x)(1-k_{10}(1)x+x^2)(1-k_{10}(2)x+x^2)(1-k_{10}(3)x+x^2)^2$$
$$= (x^9+1) + 0\cdot(x^8+x) + (1/2)(1+\sqrt{5})(x^7+x^2) + 0\cdot(x^6+x^3) + (x^5+x^4)$$
$$h_{10}(4,3;x) = (1+x)(1-k_{10}(1)x+x^2)(1-k_{10}(2)x+x^2)\cdot(1-k_{10}(4)x+x^2)^2$$
$$= (x^9+1) + 2(x^8+x) + (1/2)\cdot(5-\sqrt{5})(x^7+x^2) + (3-\sqrt{5})(x^6+x^3) + (4-\sqrt{5})(x^5+x^4).$$

Each of these polynomials has non-negative coefficients. (Note that since there are some zero coefficients, machine approximations are not to be relied on for this example.)

Let U and V be 10-sided dice whose companion polynomials are

$$p_U(x) = h_{10}(3,4;x) \qquad p_V(x) = h_{10}(4,3;x).$$

As before, $L(U+V) = L(X+Y)$, but U and V are not fair.

Note. A computer search (using MACSYMA) revealed this as the only such example for $n \leqslant 11$.

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EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

Three Old Problems about Polynomials with Real Roots

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We consider only polynomials all of whose roots are real. For such a polynomial of degree $n > 1$, the **span** is the maximum distance between two roots, and the **gap** is the minimum distance. All three problems concern spans, and the third also involves gaps.

Problem 1. The polynomials considered here are monic and lie in $\mathbb{Z}[x]$, that is, have integer coefficients and leading coefficient 1. It was shown in [**1**] that there are infinitely many monic polynomials in $\mathbb{Z}[x]$ which are irreducible and have all their roots in $[a, b]$, provided that $b - a > 4$. Pólya had proved earlier that the number is finite if $b - a < 4$. If $b - a = 4$, then there are infinitely many such polynomials if a and b are integers. Are there any other cases with $b - a = 4$ where the number is infinite? Among the monic polynomials in $\mathbb{Z}[x]$ whose roots all lie in $[-2, 2.5]$ but not all in $[-2, 2]$, are there infinitely many which are irreducible and have span < 4? If the answer to the first question is yes, then so is the answer to the second, but perhaps not conversely.

Problem 2. Here we allow polynomials in $\mathbb{R}[x]$, that is, polynomials with real coefficients. An attempt was made in [**2**] to find the maximum possible span for the kth derivative of a polynomial $f(x)$ all of whose roots lie in $[-1, 1]$. The only nontrivial cases are those with $k + 2 \leqq n \leqq 2k + 1$. It was shown that in these cases the maximum can be attained only when all the roots of $f(x)$ lie at 1 or -1. The obvious conjecture is that these roots must be distributed as equally as possible between the two end points. How can this be proved?

Problem 3. A study was made in [**3**] of monic polynomials in $\mathbb{Z}[x]$ having real roots and span < 4. The objective was to determine all such polynomials of degree $n \leqq 8$ which are irreducible. In the process, it was necessary to compute some reducible polynomials with distinct roots. It was noticed that there is a strong correlation between reducibility and the smallness of the gap. For example, among the monic polynomials in $\mathbb{Z}[x]$ with span < 4, there are 17 essentially different sextics which are irreducible, and all have gaps > 0.23. On the other hand, there are 19 such sextics which are the product of two irreducible cubics, and 12 of these have gaps < 0.15. Is there a general theorem which determines values of n, s, and g, so that a monic polynomial in $\mathbb{Z}[x]$ of degree n with real roots is reducible if it has span $< s$ and gap $< g$? Many reducible polynomials with distinct roots should satisfy the conditions. One suitable triple is $n = 6$, $s = 4$, and $g = 0.23$.

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A Challenging Definite Integral

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In spherical 3-space, that is, on the 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ in Euclidean 4-space, a certain tetrahedron of known volume (which Schläfli called an **orthoscheme**) can be dissected into three smaller orthoschemes (two of them congruent) whose volumes can be expressed in terms of Schläfli functions [**2**, pp. 177–179; **1**, pp. vii, 6–12, 195]. It follows that

$$f + 2g = 2/15,$$

where

$$f = \frac{1}{\pi^2}\int_1^6 \frac{\operatorname{arc\,sec} t\,dt}{(t+2)\sqrt{(t+1)(t+3)}}, g = \frac{1}{\pi^2}\int_1^6 \frac{\operatorname{arc\,sec} t\,dt}{(t+2)\sqrt{t+1}}.$$

N. J. A. Sloane has used a computer to show that

$$f \approx 0.02268\ 05970\ 96406\ 8,$$

$$g \approx 0.05532\ 63681\ 18463\ 3.$$

The problem is to establish the precise value of $f + 2g$ without appealing to geometry or the computer!

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How Big a Slice Can You Make Through a Cube?

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Prove or disprove that for each $n \geq 2$, the maximum $(n-1)$-dimensional volume of a cross section of an n-dimensional unit cube is $\sqrt{2}$.

LETTERS TO THE EDITOR

Editor:

In the Notes section of this MONTHLY, Aug.–Sept. 1987, pp. 662–663, Boo Rim Choe offers an elegant proof of $\Sigma 1/n^2 = \pi^2/6$. It must, however, be remarked that this proof is by no means new: In a slightly different form it had been published by Euler in 1743 ([**1**]); cf. [**2**] (p. 388; footnote to nr. 210) and [**3**]. (Stäckel's paper, mentioned in [**3**], is reprinted in Euler's *Opera Omnia* I, 14, pp. 156–176.)

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Two Discrete Forms of the Jordan Curve Theorem

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The Jordan curve theorem is one of those frustrating results in topology: it is intuitively clear but quite hard to prove. In this note we will look at two discrete analogs of the Jordan curve theorem that are easy to prove by an induction argument coupled with some geometric intuition. One of the surprises is that when we discretize the plane we get two Jordan curve theorems rather than one, a consequence of the interplay between two natural products in the category of graphs. Topology in this context has been studied by Farmer in [**2**].

To state the discrete versions, we need to know what the discrete analog of the plane is and what plays the role of a simple closed curve. Since the plane is the topological product of two lines, we take as our discrete analog the product of two discrete lines. We will use undirected graphs for our analogs of spaces, with vertices for points and edges connecting points which are to be thought of as touching.

DEFINITION 1. A discrete n point line $[1, n]$ is a graph with vertices $\{1, 2, \dots, n\}$ and edges connecting each vertex to itself and to its successor. The discrete line L is a similar graph based on all of the integers.

DEFINITION 2. A discrete n point circle is a discrete n point line with n and 1 connected by an edge.

There are two important products in the category of graphs: the categorical product and the tight product. The tight product is used in building graphs using a sort of prime factorization in Behzad and Chartrand [**1**].

DEFINITION 3. The *product* of two graphs $(V_1, E_1)\prod(V_2, E_2)$ has the set $V_1 \times V_2$ as vertices and has (v_1, v_2) connected to (v_1', v_2') by the edge (e_1, e_2) if e_1 connects v_1 and v_1' and e_2 connects v_2 and v_2'.

DEFINITION 4. The *tight product* of two graphs has $V_1 \times V_2$ as its set of vertices and has an edge connecting (v_1, v_2) and (v_1', v_2') if and only if $v_1 = v_1'$ and there is an edge connecting v_2 and v_2', or $v_2 = v_2'$ and there is an edge connecting v_1 and v_1'. We denote this as $(V_1, E_1)\square(V_2, E_2)$.

If we take the product of two lines we get a patch of the plane with points connected which are nearest neighbors vertically, horizontally, or diagonally. If we take the tight product we leave out the diagonal connections.

The analog of continuous functions will be mappings of graphs: vertices are taken to vertices and edges to edges. A closed curve is the image of a circle under a graph map. It is simple if the map also reflects adjacency; that is, if $c(v)$ has an edge connecting it with $c(v')$ then v and v' had an edge connecting them too.

Simple curves then are forbidden to touch themselves, not just forbidden to cross themselves. This puts us in a position to state the two forms of the Jordan curve theorem.

THEOREM (Jordan curve theorem for tight closed curves). *If s is a simple closed curve with domain having at least* 8 *points in $L \square L$, then $L \times L \setminus \mathrm{im}(s)$ has exactly two product path components.*

THEOREM (Jordan curve theorem for product closed curves). *If s is a simple closed curve with domain having at least* 4 *points in $L \sqcap L$, then $L \times L \setminus \mathrm{im}(s)$ has exactly two tight path components.*

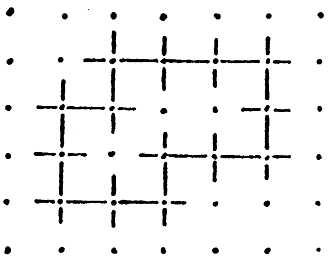

FIG. 1.

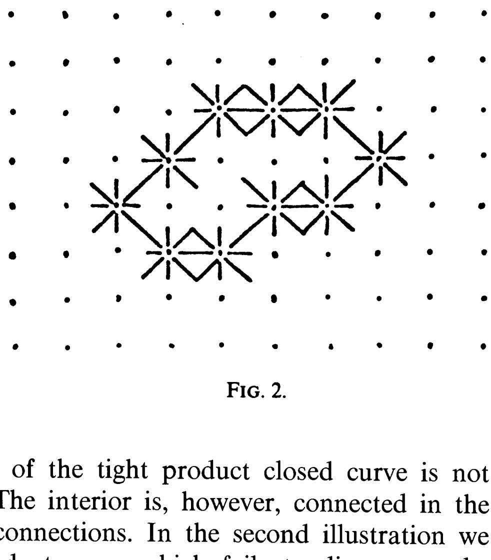

FIG. 2.

Notice that in Figure 1 the interior of the tight product closed curve is not connected in the tight product space. The interior is, however, connected in the product space, which allows diagonal connections. In the second illustration we have a simple closed curve in the product sense which fails to disconnect the categorical product space. If we use the tight product instead, then the interior is not connected to the exterior and each forms a connected set. The minimum size restriction eliminates the trivial cases in the next illustration.

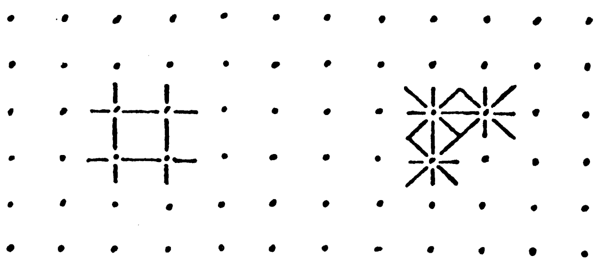

FIG. 3.

Proof (for product closed curves). Since a simple closed curve involves only a finite number of points we can move it into the first quadrant and guarantee that the coordinates of points are bigger than 0 and less than m for sufficiently large m. We define the *rank* of s as the triple (N, X, Y) where N is the number of distinct points in the closed curve and (X, Y) is the point in the closed curve with largest first coordinate X and largest second coordinate Y of the points of $\mathrm{im}(s)$ with that first coordinate. Ranks are ordered lexicographically. This is a well-ordering, so strong induction on rank is a valid proof technique.

The smallest simple closed curve for this theorem has $N = 4$. It forms a diamond surrounding a single point which forms the inside component. All other points are connected to the point $(0,0)$ by a tight path. The requirement that a simple curve reflect adjacency eliminates other possible curves of length four. Thus the theorem is true for closed curves with length 4.

Now suppose that the theorem has been proved for all closed curves with rank less than (N, X, Y) and that s is a simple closed curve with rank (N, X, Y). We will reduce the rank by moving the point (X, Y) to $(X - 1, Y)$. The points in the closed curve s which were adjacent to (X, Y) could only be among $(X, Y - 1)$, $(X - 1, Y - 1)$, and $(X - 1, Y + 1)$. (Two points are adjacent to (X, Y) and they must be nonadjacent, hence, $(X - 1, Y)$ is not one of the possible points.) All of these are adjacent to $(X - 1, Y)$ so the result is still a closed curve, though it may not be a simple closed curve. Observe that moving this point reduces the rank. If the new closed curve is a simple closed curve then we are done since the interior of the original curve is the interior of the curve of lower rank with the point $(X - 1, Y)$, which is tight adjacent to it, added. The exterior of the original curve is the exterior of the new curve with the point (X, Y) removed. This is still tight connected since any tight path passing through (X, Y) in the exterior of the lower rank curve can take a detour through $(X, Y + 1)$, $(X + 1, Y + 1)$ and $(X + 1, Y)$.

There are two ways for the resulting closed curve to fail to be simple: either the point $(X - 1, Y)$ is adjacent to one of the points two steps away from (X, Y) in s, or it is adjacent to a point more than two steps away. If (X, Y) was $s(h)$ and $(X - 1, Y)$ is adjacent to $s(h - 2)$ then we can remove $s(h - 1)$. If (X, Y) was $s(h)$ and $(X - 1, Y)$ is adjacent to $s(h + 2)$ then we can remove $s(h + 1)$. Removing these points, if necessary, will further reduce the rank. The interior of the resulting curve is tight connected to (X, Y), so the interior of the original curve is tight connected. Any tight path passing through one of the points removed has a detour which avoids them and stays in the exterior. Figure 4 shows how this works for a typical case.

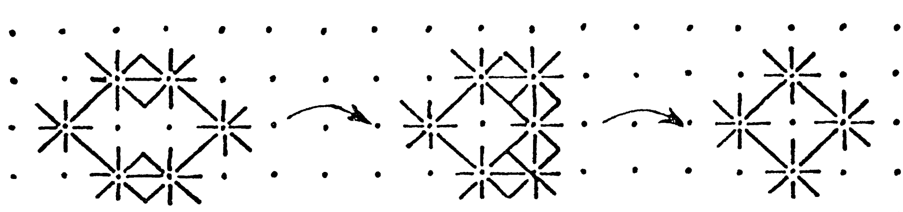

FIG. 4.

Suppose that (X, Y) is $s(h)$ and $(X - 1, Y)$ is adjacent to $s(k)$ where k is more than two away from h. Then by moving to $(X - 1, Y)$ we pinch the closed curve into two closed curves which have a tight path connecting their interiors which passes through the point $(X-1, Y)$ and each of which is strictly shorter than our original loop. (See Figure 5.) Since they have smaller ranks they each divide the product into exactly two tight pieces. The interior of s is then the union of the interiors of these two new closed curves plus the point $(X-1, Y)$. It remains to show that the exterior is tight path connected.

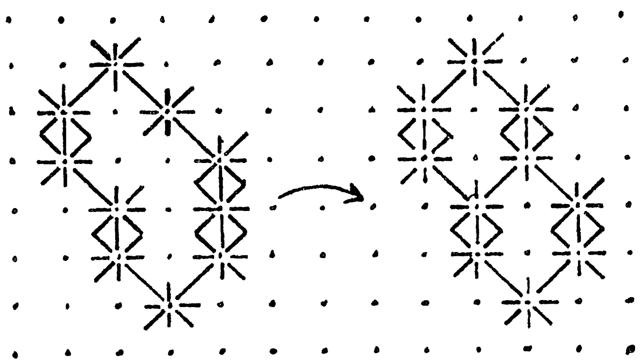

FIG. 5.

The exterior is the intersection of the exteriors of the two new closed curves. Call the new closed curves s_1 and s_2 and renumber so that the intersection points are at $t=0$ and $t=1$, with $(X-1, Y)=s_2(1)$. Let p and q be in $\operatorname{ext}(s_1) \cap \operatorname{ext}(s_2)$. Since $\operatorname{ext}(s_1)$ is tight path connected there is a tight path in $\operatorname{ext}(s_1)$ from p to q. If that path is also in $\operatorname{ext}(s_2)$ then nothing more needs to be done. If not then there are points p' and q' such that p' is the last point in the path for which the segment from p to p' is in $\operatorname{ext}(s_2)$ and q' is the first so that the segment from q' to q is in $\operatorname{ext}(s_2)$. It follows that both p' and q' are adjacent to points in s_2. Thus to prove the theorem it will suffice to show that the set E of all points adjacent to s_2 and in the exterior of both curves is tight connected.

Since the original curve was simple we know that $s_2(0)$ and $s_2(1)$ are the only points in s_2 that are adjacent to points in s_1. We will show that E is tight path connected by walking around s_2 starting at (X, Y) and observing what happens in each nine point patch with an element of s_2 at the center. It is not difficult to list all of the ways that a product path can pass through a nine-point patch (see Figure 6) and in all cases the points on either side of the path form tight path connected sets. Since s_2 is of finite length we can piece together such patches to see that the set E is tight path connected.

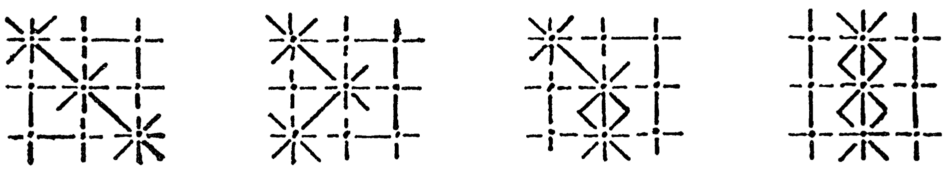

FIG. 6.

The proof of the theorem for simple closed curves in the tight product is a similar, though slightly less difficult, induction argument. The rank is defined the same way as in the product case. The reduction is done by moving the point (X, Y) to $(X - 1, Y - 1)$ which either gives a simple closed curve of lower rank or pinches the curve in two or gives a curve which can be shortened. Analysis of the possible nine-point patches again allows us to show that the exterior is connected.

The author would like to thank the referees and Randall Weiss for suggestions which improved this note.

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Some Polynomial Identities that Imply Commutativity for Rings

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1. Introduction. Johnsen, Outcalt, and Yaqub **[5]** considered the ring-theoretic analogue of a well-known group-theoretic result, which states that a group G satisfying $(xy)^2 = x^2y^2$ for all x and y in G is necessarily abelian. It is also an easy exercise to show that a group G is abelian if $(xy)^2 = yx^2y$ for all x and y in G; the corresponding ring-theory version has not yet appeared in any text. It is surprising that the ring theoretic analogue of many such results escaped the attention of the research workers, though the commutativity of the rings satisfying other identities such as $xy^2x = yx^2y$ has been considered fully **[1]**, **[3]**, **[6]**. The reason for this sort of omission is understandable. We know that the essential mechanism in the proof of such results in groups is cancellation, which is not permissible in a general class of rings. Only a few results could be proved by going through several permutations of the substitutions of, say, y by $x + y$ and x by $x + 1$ starting with the given identity. To obtain other results, complicated combinatorial arguments had to be used **[2]**, **[4]**. In this note our objective is to establish a result which allows a limited cancellation property in rings with unity. The proof depends on the simple strategy of substituting $x + 1$ for x to get another identity simpler than the original one. Indeed, we prove the following:

THEOREM A. *Let R be an associative ring with unity* 1 *and let $F(X, Y, Z)$ be a polynomial with coefficients from elements of R where the indeterminates commute neither with each other nor with the elements of R. Suppose that F is homogeneous in X of degree n and homogeneous in Y of degree m and that $F(x, y, xy - yx) = 0$ for all x and y in R. Then $m!n!F(1, 1, xy - yx) = 0$ for all x and y in R.*

In Section 3 we shall use our theorem to find some ring-theory version of the group-theoretic results including that of Johnsen, Outcalt, and Yaqub [**5**] and that of Awtar [**2**].

2. Proof of Main Theorem. Let $R[X, Y, Z]$ denote the ring of the polynomials in noncommuting indeterminates X, Y, Z over R. Define an automorphism σ on $R[X, Y, Z]$ by:

$$\sigma[F(X, Y, Z)] = F(X+1, Y, Z) \tag{1}$$

and a σ-derivation $\Delta = \sigma - I$:

$$\Delta[F(X, Y, Z)] = F(X+1, Y, Z) - F(X, Y, Z). \tag{2}$$

Easy computations show that for any two polynomials F and G in $R[X, Y, Z]$, we have

$$\Delta[F+G] = \Delta[F] + \Delta[G] \tag{3}$$

$$\Delta[FG] = (\Delta[F])(\sigma[G]) + F(\Delta[G]) \tag{4}$$

and an induction gives the Leibniz formula

$$\Delta^n[FG] = \sum_{r=0}^{n} \binom{n}{r} (\Delta^r[F])(\sigma^r\Delta^{n-r}[G]). \tag{5}$$

This allows us to prove:

LEMMA. *If F is homogeneous of degree n in X, then $\Delta^n[F(X, Y, Z)] = n!F(1, Y, Z)$ and $\Delta^m[F(X, Y, Z)] = 0$ for $m > n$.*

Proof of Lemma. By (3) it suffices to prove the lemma when $F(X, Y, Z)$ is a monomial. It can be proved by induction on n. If $n = 0$, $F(X, Y, Z)$ is independent of X and $\Delta^0[F(X, Y, Z)] = F(X, Y, Z) = F(1, Y, Z)$. Again $\Delta[F(X, Y, Z)] = F(X+1, Y, Z) - F(X, Y, Z) = 0$ and, hence, $\Delta^m[F(X, Y, Z)] = 0$ for all $m > n$.

For the induction step, write the monomial $F(X, Y, Z)$ as AXG, where A is a monomial with no X's in it, and G is a monomial of degree $n-1$ in X. Then $\Delta[A] = 0$ and by (4), $\Delta[AX] = A$; hence, $\Delta^r[AX] = \Delta^{r-1}[A] = 0$ for $r > 1$ by the case $n = 0$. Again by using (5), we get

$$\Delta^m[AXG] = (AX)\Delta^m[G] + nA(\sigma\Delta^{m-1}[G]).$$

By the induction hypothesis, the first term on the right side is zero if $m \geqslant n$. The second term is zero if $m > n$; if $m = n$, this term equals $nA(n-1)!G(1, Y, Z) = n!F(1, Y, Z)$, which proves the lemma.

Proof of Theorem A. If $F(x, y, xy - yx) = 0$ for all x and y in R, then on replacing x by $x+1$, we get

$$F(x+1, y, (x+1)y - y(x+1)) = F(x+1, y, xy - yx) = 0.$$

That is, if $F(x, y, xy - yx)$ is zero, the same is true for $\sigma[F]$, $\Delta[F]$, and $\Delta^n[F]$. Hence, $n!F(1, y, xy - yx) = 0$ for all x and y in R. By applying the whole procedure again on the polynomial $F(1, Y, Z)$, which is homogeneous of degree m in Y by using a new derivation, Δ' defined as

$$\Delta'[F(X, Y, Z)] = F(1, Y+1, Z) - F(Y, Z).$$

The result is the conclusion of the theorem.

3. Applications to Commutativity Theorems. We can derive a number of results with the help of our theorem proved above. As we have claimed in the beginning, even those results which could be proved earlier using very complicated combinatorial arguments will become corollaries of our theorem. We need just to select a suitable polynomial $F(X, Y, Z)$.

To begin with we prove the following result. Let us assume hence onward that R is an associative ring with unity 1.

PROPOSITION 1. *Let R be a ring satisfying $(xy)^2 = yx^2y$ for all x and y in R. Then R is commutative.*

Proof. Take $F(X, Y, Z) = ZXY$. Then indeed $F(x, y, xy - yx) = (xy)^2 - yx^2y = 0$. Hence, by applying Theorem A, $F(1, 1, xy - yx) = 0$, that is, $xy = yx$ and the ring R is commutative.

Similarly, by taking the polynomial $F(X, Y, Z) = XZY$ we prove the result due to Johnsen, Outcalt, and Yaqub [**5**]. As has been shown in example 3 of [**5**], if we replace the identity $(xy)^2 = x^2y^2$ by $(xy)^3 = x^3y^3$, then the commutativity is not guaranteed. The following result suggests that in this case the commutativity fails only in rings that have no 2- or 3-torsion.

PROPOSITION 2. *Let R be a ring satisfying $(xy)^3 = x^3y^3$ for all x and y in R. If 6 is not a zero divisor in R, then R is commutative.*

Proof. Take $F(X, Y, Z) = X^2ZY^2 + XZXY^2 + XYXZY$, then $F(x, y, xy - yx) = x^3y^3 - (xy)^3 = 0$ and by our Theorem A, $n!m!F(1, 1, xy - yx) = 2!2!3(xy - yx) = 0$. But 2 and 3 are not zero divisors in R, so $xy - yx = 0$, which gives commutativity.

In fact, applying Theorem A we can derive even more general results. As an example we prove below a generalization of Proposition 2 which was earlier established by Awtar [**2**].

PROPOSITION 3. *Let $n > 1$ be a positive integer and R be a ring in which no prime number $\leqslant n$ is a zero divisor. If R satisfies $(xy)^n = x^ny^n$ for all x and y in R, then R is commutative.*

Proof. Just as we did in the proof of Proposition 2, we take $F(X, Y, Z)$ to be a sum of $n(n-1)/2$ monomials each of which is a product of one Z by $n-1$ X's and $n-1$ Y's (it takes one term to move each Y in $XYXY\ldots$ to the right, past one X). Then we have $F(x, y, xy - yx) = x^ny^n - (xy)^n = 0$, and by Theorem A,

$$m!n!F(1, 1, xy - yx) = ((n-1)!)^2 \cdot (n(n-1)/2)(xy - yx) = 0.$$

This implies that $xy - yx = 0$ if all primes dividing $((n-1)!)^2(n(n-1)/2)$ are not zero divisors.

REMARKS 1. If we take $F(X, Y, Z) = XZ - 2ZY$, the polynomial identity $(x + 2y)xy = xy(x + 2y)$, or if we take $F(X, Y, Z) = XYZ$, then the polynomial identity $(xy)^2 = xy^2x$ for all x and y in R implies commutativity. Other examples can be constructed ad libitum ad infinitum.

2. More subtle commutativity theorems, which do not work for all rings with unity, also often assume polynomial identities of the form $F(x, y, xy - yx) = 0$, but with $F(1, 1, Z) = 0$.

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An Overlooked Example of Nonunique Factorization

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On the face of it, the familiar identity

$$\sin^2 t = 1 - \cos^2 t = (1 + \cos t)(1 - \cos t) \tag{1}$$

asserts that two different-looking pairs of factors have the same product. It seems to have gone unnoticed, however, that (1) is actually a valid example of nonunique factorization in an integral domain when looked at in the proper context. Its familiarity makes it a particularly attractive example to present to students encountering nonunique factorization for the first time. Just as the usual textbook examples involving integers in quadratic number fields, such as $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, show that unique factorization can fail in rings very much like the integers, the example treated here shows that it can fail in a ring very much like the ring of polynomials over a field.

Of course we have to do more than simply remark that the two sides of (1) look different. We must specify the ring we are working in, and then show that the factors $\sin t$, $1 + \cos t$, and $1 - \cos t$ are irreducible, and that $\sin t$ is not the product of one of the other factors and a unit (invertible element) of the ring.

We shall work with the real trigonometric polynomials, that is, the functions representable as finite sums of the form

$$a_0 + \sum^{k} (a_n \cos nt + b_n \sin nt) \tag{2}$$

in which the a's and b's are real numbers. Students who have seen anything of Fourier series find it natural enough to consider these functions, although they may not have seen them called trigonometric polynomials, or considered the question of whether they form a ring. The familiar Fourier coefficient formulas $a_0 = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \pi^{-1}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, and $b_n = \pi^{-1}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ for $n > 0$, show that the coefficients in (2) are uniquely determined by the function.

The *degree* of a nonzero trigonometric polynomial is defined as the largest value of n for which a_n and b_n are not both zero. The following well-known lemma shows that the trigonometric polynomials form a ring, and that degrees behave as they do for ordinary polynomials.

LEMMA. *The product of a trigonometric polynomial of degree m and one of degree n is a trigonometric polynomial of degree $m + n$.*

Proof. The assertion of the lemma is obvious if m or n is 0, because a trigonometric polynomial of degree zero is simply a constant function. From now on we assume $m, n > 0$. Recall the standard identities for expressing products of sines and cosines in terms of sums and differences of other sines and cosines:

$$(\sin a)(\sin b) = [\cos(a-b) - \cos(a+b)]/2$$

$$(\cos a)(\cos b) = [\cos(a-b) + \cos(a+b)]/2$$

$$(\sin a)(\cos b) = [\sin(a+b) + \sin(a-b)]/2.$$

Applying these to the product of $p\cos mt + q\sin mt$ and $r\cos nt + s\sin nt$ and collecting terms, gives the result

$$A\cos(m-n)t + B\sin(m-n)t + C\cos(m+n)t + D\sin(m+n)t, \qquad (3)$$

where $A = (pr + qs)/2$, $B = (ps - qr)/2$, $C = (pr - qs)/2$, and $D = (ps + qr)/2$. When $m > n$, (3) is already in the form (2). If $n > m$, replacing $\cos(m-n)$ by $\cos(n-m)$ and $\sin(m-n)$ by $-\sin(n-m)$ puts it in the proper form, while if $m = n$ it is necessary to replace $\sin 0$ by 0 and $\cos 0$ by 1. Direct calculation gives

$$C^2 + D^2 = (p^2 + q^2)(r^2 + s^2)/4,$$

which shows that if neither factor is zero (so $p^2 + q^2 \neq 0$ and $r^2 + s^2 \neq 0$) then $C^2 + D^2 \neq 0$, so the product has degree $m + n$.

Now consider the product of any two trigonometric polynomials of respective degrees m and n. It is a sum of products of terms of the type just considered, so it is a trigonometric polynomial. The product of the two high-order terms gives a non-zero term of degree $m + n$ which cannot be cancelled by any other term in the product, so the result has degree $m + n$ as claimed.

PROPOSITION. *The trigonometric polynomials form an integral domain. Furthermore,*

(a) *The units (invertible elements) in this domain are the elements of degree 0, that is, the constant functions.*
(b) *All elements of degree 1, including $\sin t, 1 + \cos t$, and $1 - \cos t$, are irreducible.*

The proposition follows at once from the lemma, just as with ordinary polynomials, and we leave the details to the reader.

It follows from (a) and (b) that the factors in (1) are irreducible and that $\sin t$ is not the product of one of the other factors with a unit. Hence we have a genuine case of nonunique factorization.

One can very well stop here in an elementary discussion, but the example does raise another point that may be of interest.

The proof of the lemma uses the fact that the sum of the squares of two real numbers is zero only when both are zero, and breaks down if complex coefficients are allowed. Using the complex exponential forms of the sine and cosine shows that the ring of trigonometric polynomials with complex coefficients is the same as the ring of polynomials in positive and negative powers of $z = e^{it}$ with complex coefficients. To see that this is a unique factorization ring, define the degree of a polynomial in z and z^{-1} as the difference between the largest and smallest exponents appearing in non-zero terms. With this definition, the elements of degree zero are the monomials, which are exactly the invertible elements in this ring. The usual proof that ordinary polynomials over a field form a Euclidean ring then goes through with no essential change.

What is it about the change of coefficients that alters the nature of factorization in the ring? For one thing, introducing complex coefficients produces many more units—all the non-zero constant multiples of powers of $z = \cos t + i\sin t$ and $z^{-1} = \cos t - i \sin t$. Our particular example breaks down because the factors involved cease to be irreducible. We have

$$\sin t = (z - z^{-1})/(2i) = z^{-1}(z-1)(z+1)/(2i),$$

$$1 - \cos t = (-z + 2 - z^{-1})/2 = -z^{-1}(z-1)^2/2,$$

and

$$1 + \cos t = z^{-1}(z+1)^2/2,$$

so both sides of (1) become

$$-z^{-2}(z-1)^2(z+1)^2/4$$

when expressed as a product of irreducible factors.

A ring of algebraic integers can sometimes be enlarged to another in a way that restores unique factorization, although the problem of how and when it can be done is not at all elementary, and as far as I know is not solved in general. For example, the ring $\mathbf{Z}[\sqrt{-3}]$ consisting of numbers of the form $a + b\sqrt{-3}$ with a and b integers does not have unique factorization, as the equation $2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$ shows. Unique factorization can be restored in this case by enlarging to the ring of *all* algebraic integers in the field $\mathbf{Q}(\sqrt{-3})$, which is $\mathbf{Z}[\omega]$, where $\omega = (1 + \sqrt{-3})/2$ is a complex cube root of one. This does not work for the ring $\mathbf{Z}[\sqrt{-5}]$ used in the example at the beginning of the paper, because that is already the ring of all algebraic integers in the field $\mathbf{Q}(\sqrt{-5})$. The ring of all algebraic integers in the enlarged field $\mathbf{Q}(\sqrt{-5}, i)$, however, which can be shown to be the ring $\mathbf{Z}[\eta]$ where $\eta = (i + \sqrt{-5})/2$ is a root of $x^4 + 3x^2 + 1$, is an enlargement of $\mathbf{Z}[\sqrt{-5}]$ that does have unique factorization. I do not know an elementary proof of the last assertion, but it is easily established by standard arguments based on Minkowski's estimate, as illustrated in [**1**, chapter 12], [**2**, chapter 13], or [**3**, chapter 5].

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Two Counterexamples in General Topology

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In [**3**] Albert Wilansky inserted two axioms between the separation axioms T_1 and T_2, namely the *US*-axiom (every convergent sequence has a unique limit) and the *KC*-axiom (every compact subset is closed). He showed that $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$ and discussed at length the problem of constructing counterexamples of compact spaces showing the failure of each of the reverse implications, proving several interesting results in the process. As a consequence of theorems 4 and 5 of [**3**], it was brought out that, if a T_2-space X is not a k-space (a k-space is one in which a set is closed iff its intersection with each closed compact set is closed) then its one point compactification X^+ is *US* but not *KC*. An example of a T_2-space, which is not a k-space is given in [**3**], example 7. Since this example involves the Cech compactification, it seems worthwhile to have the following two elementary examples.

Example 1. The Appert space A (see [**1**], p. 117) whose ground set is the set of all positive integers and $E \subseteq A$ is open iff either $1 \notin E$ or $1 \in E$ and

$$\lim_{n\to\infty} \frac{1}{n}\left\{\sum_{r=1}^{n} \chi_E(r)\right\} = 1.$$

Example 2. The Fortissimo space F (see [**1**], p. 53) whose ground set is any uncountable set with a particular point p and $E \subseteq F$ is open iff either $p \notin E$ or $p \in E$ and $F \setminus E$ is countable.

One easily verifies that both A and F above are noncompact T_2-spaces, that both are pseudofinite (i.e., all compact subsets are finite) and that neither is discrete. Hence, neither of them is a k-space so that both A^+ and F^+ are *US* but not *KC*. This can in fact be shown directly. While $A^+\setminus\{1\}$ and $F^+\setminus\{p\}$ are compact nonclosed subsets of A^+ and F^+, respectively, one easily imitates the proof of theorem 4 in [**3**] to show that they are *US* spaces.

Other examples may be found in [**2**, example 2.3] and [**4**, p. 345].

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EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

An Elementary Approach to $y'' = -y$

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THEOREM. *Every solution of* $y'' + k^2y = 0$ $(k > 0)$ *on the interval* $a < x < b$ *has the form* $y = c_1\cos kx + c_2\sin kx$, *where* c_1, c_2 *are constants.*

Proof. Without loss of generality, take $k = 1$ (the general case comes from the change of independent variable $t = kx$). Define

$$\begin{aligned} c_1 &= y\cos t - y'\sin t, \\ c_2 &= y\sin t + y'\cos t. \end{aligned} \tag{1}$$

Since $y'' + y = 0$, it is clear that $c_1' = c_2' = 0$, so that c_1, c_2 are constant. Eliminate y' from equations (1). □

There are many other proofs of this theorem. From $y'(y'' + y) = 0$, one finds $y'^2 + y^2 = r^2 = \text{const}$, so that (if $r > 0$) $dy/\sqrt{r^2 - y^2} = \pm dt$, $y = r\sin(\pm t + C)$. See also [**1**, pp. 83–84].

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A Simple Proof for the Simplicity of A_5

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Dedicated to Günther Pickert on the occasion of his 70th birthday

The alternating group A_5 is usually the first example of a simple nonabelian group for the student. The group A_5 is isomorphic to the group D of rotations of a regular dodecahedron (or icosahedron). I think it may be worthwhile to see a proof for the simplicity of D whose ideas are *immediately transparent*, *easily remembered*, and use nothing more than the concept of a group and a homomorphism between groups. Using the representation of A_5 as a rotation group, we put A_5 in perspective with the simple group SO_3. Other short proofs of the simplicity of A_5 have recently been proposed by Gallian [**4**], Laue [7], and David [**2**]. These, like the well-known standard proofs for the simplicity of A_n, are within the framework of permutation groups.

The pedagogical philosophy of this paper is in agreement with Waterhouse [**9**]. With Waterhouse I believe that the use of intuitively obvious geometrical facts is a benefit for the beginner, leaving out technicalities and directing his attention to the really important arguments of the proof.

Apart from group theory, the regular dodecahedron deserves attention in itself as one of the oldest topics of mathematics that is still alive in modern theories (see, e.g., [**1**], [**6**], [**8**]). Felix Klein's book [**5**] on the icosahedron contains the basic idea of looking at the elements of order 2, 3, and 5 of the group A_5 (or D), which is essentially the same in all proofs. From the geometric inspection of rotations of this order we will get the simplicity of D in the present note.

Let me first give the proof in a shorthand manner; the details will be supplied later. Let D be the group of rotations of the regular dodecahedron. D has 60 elements, specifically 24 elements of order 5, 20 of order 3, and 15 of order 2. Looking at the way these mappings operate on the vertices of the dodecahedron we observe the following facts:

(2a) All elements of order 2 are conjugate.
(2b) The elements of order 2 generate D.
(3a) All subgroups of order 3 are conjugate.
(3b) The elements of order 3 generate D.
(5a) All subgroups of order 5 are conjugate.
(5b) The elements of order 5 generate D.

Now let $\Phi: D \to H$ be a homomorphism of groups that is not injective. Then there must be an element x, different from the identity of D, in the kernel of Φ. From $\Phi(x) = e$ we get $\Phi(x^k) = e = \Phi(a^{-1}xa)$ for any exponent k and any $a \in D$. If x is an element of order 2, by (2a, b) we have that the whole of D is mapped onto the neutral element of H. By (3a, b) or (5a, b) the same follows if x is of order 3 or 5 and, therefore, D can have no proper homomorphism.

For a presentation of this proof in a class of beginners we need to show: (A) that the rotation groups of the dodecahedron consists of no more than the 60 elements listed above, and (B) the facts (2a)–(5b). When this is done we could use some more concepts of group theory to establish the isomorphism between D and A_5 and finally elaborate some details with elementary linear algebra.

(A) Certain special projections of the dodecahedron will make the symmetries of order 2, 3, and 5 obvious:

Since there are 15 pairs of parallel opposite edges, we get 15 rotations of order 2 as in Fig. 1(2). Similarly we find 10 axes for 20 rotations of order 3 from FIG. 1(3) and 6 axes for 24 rotations of order 5 from FIG. 1(5). Together with the identity we have 60 rotations in all. Looking at the vertices A, B, C in Fig. 2 and their images under a rotation ρ, we see that there can be no more than 60 rotations in all. Finally

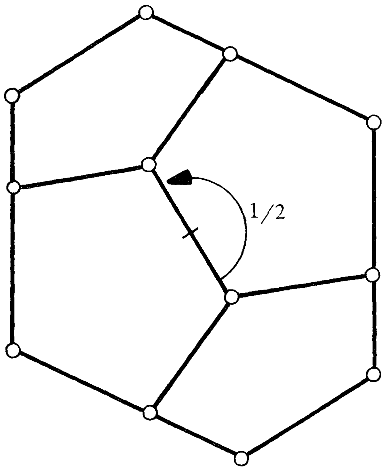

FIG. 1(2)

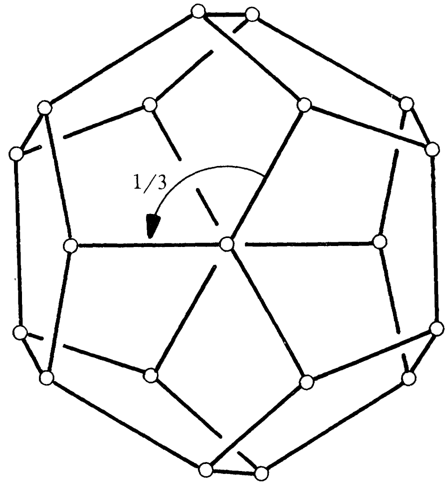

FIG. 1(3)

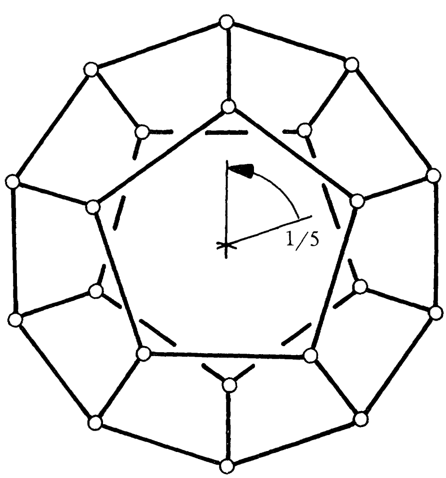

FIG. 1(5)

we may observe that the order of a rotation is determined by the type of its axis in the dodecahedron.

(B) Let R be the center of the face (pentagon) ABCDE of the dodecahedron, let r be the line OR, and let $\rho = \mathrm{rot}(r, 1/5)$, the rotation with axis r sending A $\to$ B $\to$ C etc. Similarly let S be the center of BCHMG and $\sigma = \mathrm{rot}(s, 1/5)$ with C $\to$ B $\to$ G

$\to$ M etc. Composing ρ and σ we get

$$\begin{array}{ccccccccc} & A & B & C & D & \dots & F & G \\ \rho: & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ & B & C & D & E & & G & H \\ \sigma: & \downarrow & \downarrow & \downarrow & & & & \downarrow \\ & G & B & A & & \dots & & C \end{array} \qquad \begin{array}{cccccc} & A & B & C & G & \dots \\ \sigma\rho: & \downarrow & \downarrow & \downarrow & \downarrow & \\ & G & B & A & C & \dots \end{array}$$

$\sigma\rho = \beta = \mathrm{rot}(b, \frac{1}{3})$ is the $\frac{1}{3}$-rotation with axis $b = OB$ and angle $2\pi/3$. Similarly, if we take $a =$ OA and $\alpha = \mathrm{rot}(a, 1/3)$ we get $\alpha\beta = \mathrm{rot}(r, 3/5) = \rho^3$. (Students should calculate this and the product $\beta^{-1}\rho\beta$ as above.)

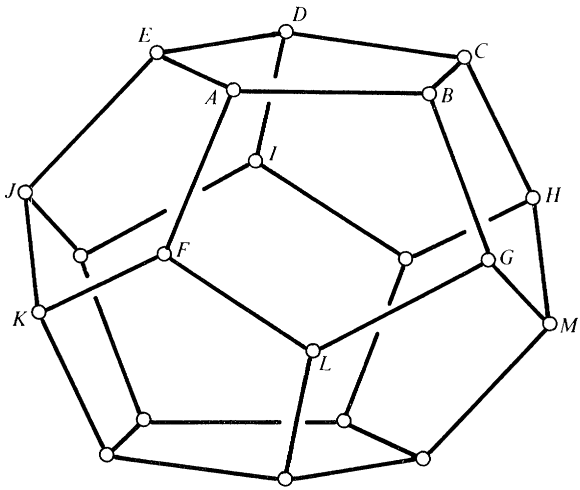

FIG. 2

Let us now prove the claims (5a, b). For (a) we take ρ, β as before and obtain $\beta^{-1}\rho\beta = \sigma$. As easy consequences we have $\beta^{-1}\rho^2\beta = \sigma^2$ and so on, and likewise we get every other rotation of order 5 from ρ. (We need not speak about conjugate subgroups as is done in (5a).) The students should note that, in geometrical terms, we use β to move the axis r to s. This underlying principle makes the calculations obvious.

(5b) We have already seen $\sigma\rho = \beta$, an element of order 3. Since the geometrical situation is the same around every vertex, we get every other element of order 3 similarly. Let Z be the center of the edge [AB]. How can we obtain the half-turn ζ about the axis OZ? In order to turn the arrow $\overline{AB}$ around we first move it by ρ into position $\overline{BC}$ and then turn it back with $\beta = \sigma\rho$. Computing $\beta\rho = \sigma\rho^2$ we get $\beta\rho = \sigma\rho^2 = \zeta$. Again from the geometrical situation it is obvious that we can generate all other half-turns in the same way.

(2a, b) and (3a, b): Meanwhile the students will have realized that it is worthwhile to look at the axes of the rotations under consideration. By moving the axis back and forth in an appropriate way we prove the claims (2a) and (3a) as was done for (5a).

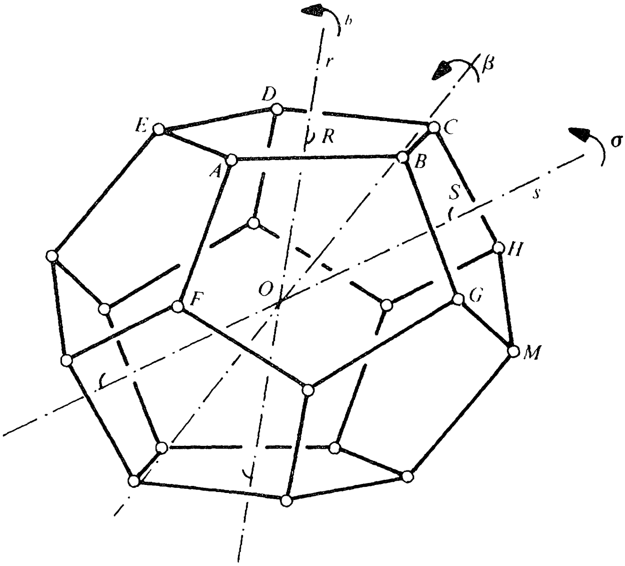

FIG. 3

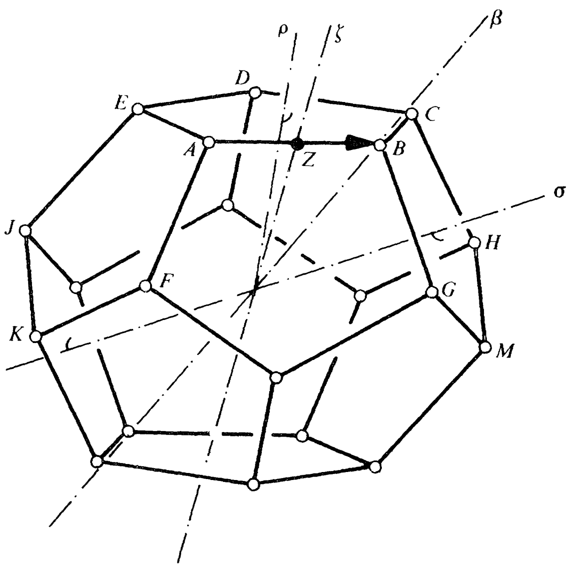

FIG. 4

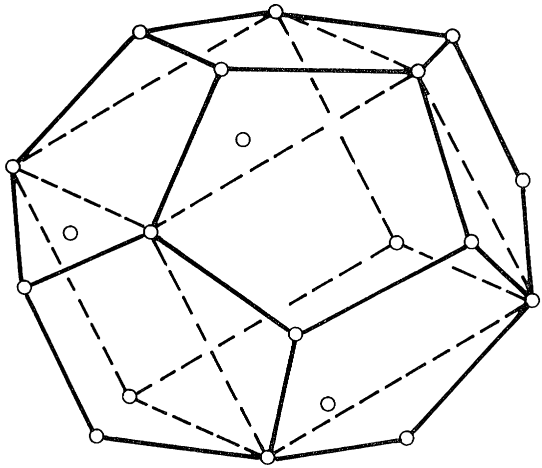

FIG. 5

For (2b) we should try to move the dodecahedron by half-turns so that, for instance, point A comes back to its original position. The result will be—if not the identity—a rotation of order 3. Similarly for rotations of order 5 and for (3b).

(C) Inscribed in the regular dodecahedron we find 5 cubes; one of them is indicated in Fig. 5. In fact, Euclid already made use of these cubes in his construction of the dodecahedron in the *Elements* XIII, 17 (see [**3**]). The edges of the cubes are diagonals of the faces of the dodecahedron, and the 12 edges of a cube are distributed over the 12 faces of the dodecahedron. If we select a specific face of the dodecahedron, the five cubes $C_1, \ldots, C_5$ will be determined by the five diagonals of that face. It is now easily seen (and in fact well known), that D induces all permutations of A_5 of the five cubes.

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PROBLEMS AND SOLUTIONS

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For instructions about submitting solutions of Problems, which should be mailed before August 31, 1988, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

E 3259. *Proposed by Jordi Dou, Barcelona, Spain.*

Let R be a semicircular region bounded by a line L and a semicircle S with center on L. Suppose P_1 and P_2 are given points in the interior of R. We wish to find parallel lines l_1, l_2 through P_1, P_2, respectively, such that

$$\frac{P_1C_1}{P_1D_1} = \frac{P_2C_2}{P_2D_2},$$

where C_1, D_1 are the intersections of l_1 with S and L and C_2, D_2 are the intersections of l_2 with S and L. Give a necessary and sufficient condition on P_1, P_2 for such parallel lines to exist.

E 3260. *Proposed by Peter Andrews and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

In how many ways can n white squares and n black squares be chosen from a $2n$ by $2n$ chessboard in such a way that no two of the chosen squares lie in the same row or the same column?

E 3261. *Proposed by Detlef Laugwitz, Technische Hochschule, Darmstadt, West Germany.*

Let G be the group of 3 by 3 orthogonal matrices with rational entries. Let S be the subset of G consisting of the six 3 by 3 permutation matrices and all matrices of the form

$$\begin{pmatrix} \frac{a}{c} & \frac{b}{c} & 0 \\ -\frac{b}{c} & \frac{a}{c} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are integers with $a^2 + b^2 = c^2 > 0$. Does S generate G?

E 3262. *Proposed by R. G. E. Pinch, Emmanuel College, Cambridge and C. Boyd, University of Edinburgh.*

It is known that every natural number n can be expressed as the sum of four squares of integers, and that three squares suffice unless n is of the form $4^a(8b+7)$. Show that every sufficiently large natural number can be expressed as the sum of at most five squares of composite numbers (i.e., squares of positive integers divisible either by the square of a prime or by the product of two distinct primes).

E 3263. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $n \geqslant 2$ let

$$H_n = \{X = (x_1, \ldots, x_n) \colon x_1 + \cdots + x_n = 1, \quad x_1 > 0, \ldots, x_n > 0\}.$$

For $1 \leqslant k \leqslant n$ and $X \in H_n$ let

$$S_k(X) = \sum \frac{x_{i_1}}{1 - x_{i_1}} \cdots \frac{x_{i_k}}{1 - x_{i_k}},$$

where the sum extends over all k-tuples $(i_1, \ldots, i_k)$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Put

$$M_k(n) = \sup_{X \in H_n} S_k(X).$$

It is not hard to see that $M_n(n) = (n-1)^{-n}$ (D. S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, (3.2.46) on p. 214).

(a) Show that $M_2(n) = 1$.

(b) Show that $M_3(4) = 4/27$.

*(c) For what pairs k, n with $3 \leqslant k \leqslant n$ is it true that $M_k(n) = \binom{n}{k}(n-1)^{-k}$?

E 3264. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp, Littleton, MA.*

Let P_n/Q_n be the nth convergent for the continued fraction

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\cdots,$$

i.e., let $P_1 = 1$, $Q_1 = 1$, $P_2 = 2$, $Q_2 = 3$, and

$$P_n = nP_{n-1} + P_{n-2}, \qquad Q_n = nQ_{n-1} + Q_{n-2} \qquad (n \geqslant 3).$$

(Cf. Hardy and Wright, An Introduction to the Theory of Numbers, Chapter 10.)
Give asymptotic estimates for P_n and Q_n.

SOLUTIONS OF ELEMENTARY PROBLEMS

Maximal Minimal Card Shufflings

E 3143 [1986, 299]. *Proposed by Allen J. Schwenk, Western Michigan University, Kalamazoo, MI.*

A riffle shuffle of a deck of cards is the commonly used technique of cutting the deck into two portions (not necessarily equal), then, elevating the corners slightly, allowing each portion to fall card by card (not necessarily alternating) merging with the other portion, and finally pushing them together to reconstitute the pack. Given a deck of n cards in arbitrarily permuted order π, determine as a function of π the minimum number of riffle shuffles that could possibly produce the identity sequence $1, 2, \ldots, n$. Describe a procedure that attains this minimum. Which original sequences require the most shuffles?

Solution by the proposer. For the permutation $\pi = a_1, \ldots, a_n$, let a *descent* be a position j such that $a_j > a_{j+1}$. By convention, we always consider position n to be a descent; thus the identity permutation is the only sequence with 1 descent. We show that the minimum number of shuffles required is $\lceil \log_2 d(\pi) \rceil$, where $d(\pi)$ is the total number of descents in π.

By partitioning the sequence π at the descent positions, we may view π as consisting of $d = d(\pi)$ blocks $B_1, \ldots, B_d$ that are strictly increasing. Now perform a riffle shuffle by cutting the deck after the block $B_{\lceil d/2 \rceil}$ and merging the two portions so that block B_j merges with block $B_{\lceil d/2 \rceil + j}$ to form a new increasing block C_j, for each $j \leqslant \lfloor d/2 \rfloor$. If d is odd, the last block $B_{\lceil d/2 \rceil}$ is unaltered. Thus, one shuffle transforms π to a sequence with $\lceil d/2 \rceil$ descents, and iterating this procedure produces the identity permutation after $\lceil \log_2 d(\pi) \rceil$ shuffles.

On the other hand, any cut of the original deck leaves at least $\lceil d/2 \rceil$ descents in one portion of the deck, and merging cannot reduce this number. By induction on d, at least $\lceil \log_2 d(\pi) \rceil$ shuffles are needed to complete the job, so $\lceil \log_2 d(\pi) \rceil$ are needed for a sequence with d descents.

The maximum required number of shuffles is $k = \lceil \log_2 n \rceil$, and the permutations requiring this many shuffles are those with more than 2^{k-1} descents.

Also solved by K. Schilling and by L. Szuecs.

Generating Sets in a Topological Space

E 3144 [1986, 299]. *Proposed by Edwin Buchman, California State University, Fullerton, CA.*

Determine the maximum number of sets in a topological space which can be generated from one set by application (possibly repeated, in any order) of the operations of taking the complement, interior, and boundary of a set.

Solution by Jesús Ferrer, Cami Collado, Oliva (Valencia), Spain. At most 34 different sets can be constructed from one set by repeatedly taking the complement, interior, or boundary. Let A', A^0, A^* denote the complement, interior, and boundary of a set A. Then the 34 sets that can be constructed appear in the following tree.

A	A^*	A^{**}	$A^{**\prime}$				
		A^{*0}	A^{*0*}	$A^{*0*\prime}$			
			$A^{*0\prime}$	$A^{*0\prime 0}$	$A^{*0\prime 0\prime}$		
		$A^{*\prime}$					
	A^0	A^{0*}	$A^{0*\prime}$				
		$A^{0\prime}$	$A^{0\prime 0}$	$A^{0\prime 0*}$	$A^{0\prime 0*\prime}$		
				$A^{0\prime 0\prime}$	$A^{0\prime 0\prime 0}$	$A^{0\prime 0\prime 0\prime}$	
	A'	$A^{\prime 0}$	$A^{\prime 0*}$	$A^{\prime 0*0}$	$A^{\prime 0*0\prime}$		
				$A^{\prime 0*\prime}$			
			$A^{\prime 0\prime}$	$A^{\prime 0\prime 0}$	$A^{\prime 0\prime 0*}$	$A^{\prime 0\prime 0*\prime}$	
					$A^{\prime 0\prime 0\prime}$	$A^{\prime 0\prime 0\prime 0}$	$A^{\prime 0\prime 0\prime 0\prime}$

The lack of additional sets follows from $A^* = A^{\prime 0\prime} \cap A^{0\prime}$, $A^{\prime *} = A^*$, $A^{*\prime 0} = A^{*\prime}$, $A^{**0} = \emptyset = A^{0*0}$, $A^{0**} = A^{0*}$, $A^{\prime 0\prime 0\prime 0} = A^{\prime 0\prime 0}$, $A^{0\prime 0\prime 0\prime 0} = A^{0\prime 0}$, $A^{\prime 0\prime 0\prime 0*} = A^{\prime 0\prime 0*}$, $A^{0\prime 0\prime 0*} = A^{0\prime 0*}$, $A^{*0\prime 0\prime 0} = A^{*0}$, $A^{*0\prime 0*} = A^{*0*}$, $A^{***} = A^{**}$. Also $A^{\prime 0*0} = \emptyset$ and $A^{\prime 0*0\prime} = X$.

To see that the number 34 cannot be reduced, let X be the real numbers with the usual topology, and let $\mathbb{Q}$ be the rationals. Then all 34 sets are distinct when A is the set

$$A = [0,1) \cup (1,2) \cup \{3\} \cup (\mathbb{Q} \cap [4,\infty)).$$

Editorial comment. This problem is very similar to Advanced Problem 5996, [1974, 1034; 1978, 283]. That problem asks the same question for {closure, interior, union}, the answer being that at most 13 distinct sets can be constructed (see also Closure, interior, and union in finite topological spaces, *Colloq. Math.*, 38 (1977) 41–51, by L. E. Moser). Kuratowski showed that at most 14 distinct sets can be constructed using {closure, interior, complement} (see Sur l'opération $\overline{A}$ de l'analyses situs, *Fund. Math.*, 3 (1922) 182–199.

Also solved by M. Bowron, J. Hook, O. P. Lossers (Netherlands), O. Matouš (Czechoslovakia), W. D. McIntosh, L. F. Meyers, R. Patenaude, J. P. Robertson, and the proposer.

An Integral of Cosines

E 3145 [1986, 299]. *Proposed by Clinton J. Kolaski, University of Minnesota, Duluth.*

Show that

$$\int_0^{\pi} \frac{\cos nx - \cos ny}{\cos x - \cos y}\, dx = \pi \frac{\sin ny}{\sin y} \qquad (n = 0, 1, 2, \dots).$$

Solution I by W. O. Egerland and C. E. Hansen, University of Baltimore. Put

$$f_n(x, y) = \frac{\cos nx - \cos ny}{\cos x - \cos y}$$

for $n = 0, 1, 2, \dots$ and $(x, y) \in R^2$. Since there is a polynomial T_n of degree n such that $\cos n\theta = T_n(\cos \theta)$, it follows that $f_n(x, y)$ may be expressed as a polynomial in $\cos x$ and $\cos y$ of degree $n - 1$. Clearly $f_0(x, y) = 0$, $f_1(x, y) = 1$, $f_2(x, y) = 2\cos y + 2\cos x$. The addition formula for the cosine immediately gives

$$\begin{aligned} f_{n+1}(x, y) + f_{n-1}(x, y) &= \frac{2\cos nx \cos x - 2\cos ny \cos y}{\cos x - \cos y} \\ &= 2\cos nx + 2 f_n(x, y)\cos y \end{aligned} \qquad (1)$$

for $n = 1, 2, \dots$. In view of the identity

$$\sin(r + 1)y + \sin(r - 1)y = 2 \sin ry \cos y,$$

a straightforward induction argument shows that the recurrence (1) and the above initial values of $f_n(x, y)$ imply the following explicit formula

$$f_n(x, y) = \frac{\sin ny}{\sin y} + 2 \sum_{k=1}^{n-1} \cos kx \frac{\sin(n - k)y}{\sin y} \qquad (n = 2, 3, \dots). \qquad (2)$$

Thus for fixed y we have the indefinite integral

$$\int f_n(x, y)\, dx = x \frac{\sin ny}{\sin y} + 2 \sum_{k=1}^{n-1} \frac{\sin kx}{k} \frac{\sin(n - k)y}{\sin y} \qquad (n = 2, 3, \dots). \quad (3)$$

The result of the problem follows.

Solution II by Kwang Kyu Park, Korea Advanced Institute of Science and Technology, Seoul, Korea. The following formulas are well known

$$\sum_{n=0}^{\infty} r^n \cos nx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2} \qquad (|r| < 1), \qquad (4)$$

$$\int_0^{\pi} \frac{dx}{1 - 2r\cos x + r^2} = \frac{\pi}{1 - r^2} \qquad (|r| < 1), \qquad (5)$$

$$\sum_{n=0}^{\infty} r^n \sin ny = \frac{r \sin y}{1 - 2\cos y + r^2} \qquad (|r| < 1). \qquad (6)$$

Since

$$\frac{\cos nx - \cos ny}{\cos x - \cos y} \leqslant n^2$$

for $(x, y) \in R^2$, the series

$$\sum_{n=0}^{\infty} \frac{\cos nx - \cos ny}{\cos x - \cos y} r^n$$

converges uniformly in x, y provided $|r| < 1$. Thus, if $|r| < 1$, we have by (4)

$$\begin{aligned}
&\sum_{n=0}^{\infty} r^n \int_0^{\pi} \frac{\cos nx - \cos ny}{\cos x - \cos y}\, dx \\
&= \int_0^{\pi} \sum_{n=0}^{\infty} r^n \frac{\cos nx - \cos ny}{\cos x - \cos y}\, dx \\
&= \int_0^{\pi} \left\{ \frac{1 - r\cos x}{1 - 2r\cos x + r^2} - \frac{1 - r\cos y}{1 - 2r\cos y + r^2} \right\} \frac{dx}{\cos x - \cos y} \\
&= \int_0^{\pi} \frac{r - r^3}{1 - 2r\cos y + r^2} \frac{dx}{1 - 2r\cos x + r^2}.
\end{aligned}$$

Using (5) and (6) in turn, we obtain

$$\sum_{n=0}^{\infty} r^n \int_0^{\pi} \frac{\cos nx - \cos ny}{\cos x - \cos y}\, dx = \frac{r - r^3}{1 - 2r\cos y + r^2} \frac{\pi}{1 - r^2} = \sum_{n=0}^{\infty} \pi \frac{\sin ny}{\sin y} r^n.$$

Comparing coefficients of r^n, we get the desired formula.

Solution III by Kee-wai Lau, Hong Kong. Denote the integral of the problem by I_n. Substituting $z = e^{ix}$ we have

$$2I_n = \int_{-\pi}^{\pi} \frac{\cos nx - \cos ny}{\cos x - \cos y}\, dx = -i \int_c \frac{z^n + z^{-n} - \left(e^{iny} + e^{-iny}\right)}{z + z^{-1} - \left(e^{iy} + e^{-iy}\right)} \frac{dz}{z},$$

where c is the positive orientation of the unit circle. It follows that

$$\begin{aligned}
2I_n &= -i \int_c \frac{\left(z^n - e^{iny}\right)\left(z^n - e^{-iny}\right)}{\left(z - e^{iy}\right)\left(z - e^{-iy}\right)} \frac{dz}{z^n} \\
&= -i \int_c \sum_{k=1}^{n} z^{n-k} e^{i(k-1)y} \sum_{r=1}^{n} z^{n-r} e^{-i(r-1)y} z^{-n}\, dz.
\end{aligned}$$

The coefficient of z^{-1} in the last integrand is

$$\sum_{k=1}^{n} e^{i(k-1)y} e^{-i(n-k)y} = e^{-(n+1)iy} \sum_{k=1}^{n} e^{2kiy}.$$

By the residue theorem $I_n = \pi \sin ny / \sin y$ if y is not an integral multiple of π and

$I_n = n\pi(-1)^{m(n+1)}$ if $y = m\pi$. This calculation could also be carried out in terms of the original variable of integration x, using the orthogonality of the functions $e^{inx}(n = 0, \pm 1, \pm 2, \ldots)$ over $[-\pi, \pi]$ instead of the residue theorem.

Editorial Comment. S. K. Ntouyan and Hang-Fai Yeung (Australia) each pointed out that the indefinite integral (3) occurs as a problem on page 205 of Joseph Edwards, *A Treatise of the Integral Calculus*, Volume 1, Macmillan, London, 1921. Edwards attributes (3) to Hermite.

Integrating (1) (with respect to x) over $[-\pi, \pi]$ gives

$$I_{n+1} - 2I_n \cos y + I_{n-1} = 0.$$

Most solvers based their solutions on this recurrence formula and the initial values $I_0 = 0$ and $I_1 = \pi$, using either mathematical induction or the theory of second-order linear difference equations with constant coefficients. M. S. Klamkin, Morris Morduchow, and R. A. Struble observed that this method of solving the problem occurs explicitly on pages 92–93 of H. Glauert, *The Elements of Aerofoil and Airscrew Theory*, Cambridge University Press, Cambridge, 1947, and on page 80 of L. M. Milne-Thomson, *Theoretical Aerodynamics*, Macmillan, London, 1952.

If λ is any positive real number and $0 < y < \pi$, the quotient

$$\frac{\cos \lambda x - \cos \lambda y}{\cos x - \cos y} = \frac{\sin\{\lambda(x+y)/2\}\sin\{\lambda(x-y)/2\}}{\sin\{(x+y)/2\}\sin\{(x-y)/2\}}$$

is bounded for $0 \leqslant x \leqslant \pi$ and thus the integral

$$I_\lambda = \int_0^\pi \frac{\cos \lambda x - \cos \lambda y}{\cos x - \cos y}\, dx$$

exists. In *Siam Review* 24 (1982), 83–85, Solution of Problem 81-5, J. A. Boa shows that

$$I_\lambda = \frac{\pi \sin \lambda y}{\sin y} - \frac{2 \sin \lambda\pi}{\sin y} \sum_{n=0}^{\infty} \frac{(-1)^n \sin(n+1)y}{n+\lambda+1}.$$

If λ is rational, the series here can be summed in finite form. Cf. also *Siam Review*, 29 (1987) 303–305, Solution of Problem 86-10.

Solved also by 35 other readers and the proposer.

The Countable Co-countable Algebra

E 3147 [1986, 400]. *Proposed by A. Wilansky, Lehigh University, Bethlehem, PA.*

Let (X, B, μ) be a measure space such that all singletons are measurable. Let $f(x) = \mu(\{x\})$. Must f be measurable?

Solution by G. Turnwald, Mathematisches Institut der Universität, Tübingen, West Germany. The answer is No. Let X be the disjoint union of uncountable sets X_1 and X_2, and let B be the σ-algebra of subsets A of X such that A or $X - A$ is countable. Let $\mu(A) = |A \cap X_1|$, if $A \cap X_1$ is finite, and otherwise $\mu(A) = \infty$. Since $f(x) = 1$ on X_1 and $f(x) = 0$ on X_2, $f^{-1}(1)$ and $f^{-1}(0)$ are uncountable. Hence f is not measurable.

Editorial comment. Several solvers noted that the answer is Yes if $\mu(X)$ is finite, since then $f(x) = 0$ except on a countable set, which implies the measurability of f.

Also solved by J. Ferrer (Spain), J. A. Goldstein, E. Hertz, A. A. Jagers (Netherlands), D. H. King, R. Levy, J. Ling and P. G. Walsh (Canada), O. P. Lossers (Netherlands), L. A. Lucas, D. Neuenschwander (student, Switzerland), Oxford Running Club (University of Mississippi), V. Pambuccian (Romania), E. Posti (Finland), D. Ramachandran, R. H. Scissors, A. V. Stanoyevitch, J. C. Tripp, and the proposer.

Matching Socks

E 3148 [1986, 400]. *Proposed by Rick Luttmann, Sonoma State University, Rohnert Park, CA.*

Let n distinct pairs of socks be put into the laundry. (It is assumed that each of the $2n$ socks has precisely one mate.) When the laundry is returned, the socks are drawn out one at a time. Each is matched with its mate, if the mate has previously been drawn. Find a formula for the expected number $E(k)$ of pairs formed after k socks have been drawn.

Solution I by K. Grünbaum and S. Pedersen, Copenhagen V., Denmark. The probability that the ith and jth socks form a matching pair is $1/(2n-1)$. For $1 \leqslant i < j \leqslant k$, let $X_{ij} = 1$ if they match, else $X_{ij} = 0$. Then the number of pairs drawn is $\sum_{1 \leqslant i < j \leqslant k} X_{ij}$, so the expectation $E(k)$ is $E(\sum X_{ij}) = \sum E(X_{ij}) = \binom{k}{2}/(2n-1)$.

Solution II by Mark Bowron, Lynnwood, WA. The probability that pair i is present among the first k socks is $\binom{2n-2}{k-2}\Big/\binom{2n}{k}$. Let $Y_i = 1$ if pair i is present, else $Y_i = 0$. Then the number of pairs present is $\sum_{i=1}^{n} Y_i$, and the expectation is

$$E\left(\sum Y_i\right) = \sum E(Y_i) = n\binom{2n-2}{k-2}\Big/\binom{2n}{k} = \tfrac{1}{2}k(k-1)/(2n-1).$$

Solution III by Richard A. Groeneveld, Iowa State University, Ames. Let X_k be the number of pairs contained in the first k socks. Let $Y_{k+1} = 1$ if the $k+1$st sock matches a previously drawn sock, else $Y_{k+1} = 0$. The conditional probability is $\Pr(Y_{k+1} = 1 | X_k) = (k - 2X_k)/(2n-k)$. The conditional expectation is $E(X_{k+1}|X_k) = E(X_k + Y_{k+1}|X_k) = X_k + (k-2X_k)/(2n-k)$, which implies $E(k+1) = E(k) + (k - 2E(k))/(2n-k)$. The recurrence, with initial condition $E(1) = 0$, is easily solved to obtain $E(k) = \frac{1}{2}k(k-1)/(2n-1)$.

Editorial comments. Several readers computed the full probability distribution for the number of pairs present among the first k socks and then computed the expectation of that directly. Several readers generalized the problem to r-legged beings, for $r > 2$. More generally, if we have n sets of socks from creatures with $a_1, a_2, \ldots, a_n$ legs respectively, D. E. Knuth remarked that the expected number of complete sets of matching socks after k socks have been drawn at random is

$$\sum_{j=1}^{n} \binom{k}{a_j}\Big/\binom{s}{a_j},$$

where $s = a_1 + a_2 + \cdots + a_n$. Many commented that this problem illustrates the power of the linearity of the expectation over dependent random variables, as used in all solutions above. R. W. van der Waall

(Netherlands) pointed out that for even k the problem (with three solutions) appeared in Dutch in Nieuw Archief voor Wiskunde, Series 3, 15 (1967), 86–87 (problem 123).

Solved by 42 readers and the proposer. Partially solved by two others.

An Exponential Inequality

E 3151 [1986, 401]. *Proposed by Peter Ivady, Institute for Economy and Organization, Budapest, Hungary.*

Let $x \geqslant 0$, $x \neq 1$, $\lambda \geqslant 1$ and $0 \leqslant \beta \leqslant 2$ be real numbers. Prove that

$$\left(\frac{x^{\lambda}-1}{x-1}\right)^{\beta} \leqslant \lambda\left(\frac{x^{\lambda\beta}-1}{x^{\beta}-1}\right).$$

Solution by M. S. Klamkin, University of Alberta. Replacing x by $1/x$ leaves the inequality unchanged, so it suffices to consider only $x > 1$ (it is trivial for $x = 0$). Because $(e^{2\lambda at}-1)/(e^{2at}-1) = (e^{\lambda at}/e^{at}) \cdot (\sinh \lambda at/\sinh at)$, the hyperbolic substitution $x = e^{2t}$ converts the inequality to

$$\lambda\frac{\sinh \lambda\beta t}{\sinh^{\beta}\lambda t} \geqslant \frac{\sinh \beta t}{\sinh^{\beta} t} \tag{1}$$

for $t > 0$, $\lambda \geqslant 1$, and $2 \geqslant \beta \geqslant 0$.

Equation (1) holds with equality for $\lambda = 1$, so it suffices to show that the left side is a nondecreasing function of λ, or equivalently that its logarithmic derivative with respect to λ is non-negative, i.e., $(1/\lambda) + \beta t \coth \lambda\beta t - \beta t \coth \lambda t \geqslant 0$. By multiplying through by $\lambda \sinh \lambda t \cdot \sinh \lambda\beta t$ and using the addition formula for sinh, we transform this inequality into

$$\sinh \lambda t \sinh \lambda\beta t \geqslant \lambda\beta t \sinh \lambda t(\beta - 1). \tag{2}$$

Since sinh is negative for negative arguments, (2) holds for $1 \geqslant \beta \geqslant 0$, and we need only consider $2 \geqslant \beta \geqslant 1$. At $\beta = 2$, (2) reduces to $(\sinh \lambda t)(\sinh 2\lambda t - 2\lambda t) \geqslant 0$, which follows from $\sinh y \geqslant y$ for $y \geqslant 0$. To establish (2) for $2 > \beta \geqslant 1$, it suffices to show that the logarithmic derivative of the left side with respect to β is less than that of the right side. This reduces to showing

$$\frac{\lambda t}{\tanh \lambda\beta t} \leqslant \frac{1}{\beta} + \frac{\lambda t}{\tanh \lambda t(\beta - 1)}.$$

This follows immediately from the fact that tanh is an increasing function.

Also solved by E. Grosswald, V. Pambuccian (Romania), R. E. Shafer, and the proposer.

Partitioning a Collection of Infinite Sets

E 3152 [1986, 401]. *Proposed by Leopoldo Nachbin, University of Rochester, Rochester, NY.*

Let S be a collection of infinite sets. Consider the following partition property (PTP): For every $X \in S$, infinite subsets $Y_X \subset X$ can be assigned such that $Y_{X_1} \cap Y_{X_2} = \varnothing$ if $X_1, X_2 \in S$ are distinct. Prove that

1) S has property (PTP) if it is countable;

2) For every uncountable cardinal number N, there is some S whose power is N, but which fails to have property (PTP).

Solution by Kenneth Schilling, University of Michigan, Flint. (1) Suppose that $S = \{X_i\colon i \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the natural numbers. Let $(p_i\colon i \in \mathbb{N})$ be an enumeration of $\mathbb{N} \times \mathbb{N}$. We now choose elements $y_i \in U_j X_j$ recursively. If $p_i = (m, n)$, let y_i be any element of X_m other than $y_0, y_1, \ldots, y_{i-1}$. Finally, for $m \in \mathbb{N}$, let $Y_{X_m} = \{y_i\colon p_i = (m, n)$ for some $n \in \mathbb{N}\}$. Since p_i runs through all of $\mathbb{N} \times \mathbb{N}$, each set Y_{X_m} is countably infinite, and by construction the sets Y_X are disjoint for distinct $X_i \in S$.

(2) Let S be any uncountable collection of infinite sets, uncountably many of which are subsets of $\mathbb{N}$. Then if sets Y_X, $X \in S$, exist as stated in the problem (even if the condition "Y_X infinite" is weakened to "Y_X nonempty"), then $\cup\{Y_X\colon X \subseteq \mathbb{N}, X \in S\}$ must be uncountable. Since this set is a subset of $\mathbb{N}$, this is impossible.

Editorial comment. Problems like (1) are discussed in great generality by P. Erdős, F. Galvin, and R. Rado in "Transversals and multitransversals", *J. London Math. Soc.* (2) 20 (1979) 387–395. See also W. Sierpiński, *Cardinal and Ordinal Numbers*, p. 459. Assertion (2) is equivalent to a restatement of the theorem on p. 95 of Paul Alexandroff et Paul Urysohn, "Mémoire sur les espaces topologiques compacts," *Verhandelingen Kon. Adad. Wetensch. Amsterdam*, 14 (1929) No. 1.

Also solved by R. E. Bernstein, R. Gilmer, S. Gudder and J. Hagler, Humboldt State Univ. Problem Group, T. Jager, R. Levy, O. P. Lossers (Netherlands), O. Matouš (Czechoslovakia), E. Mendelson, Univ. of Newcastle Problem Solving Class (Australia), V. Pambuccian (Romania), A. K. Wayman and L. Janos, and the referee.

ADVANCED PROBLEMS

For instructions about submitting solutions of Problems, which should be mailed before August 31, 1988, see the inside front cover. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label if an acknowledgment is desired.

6570. *Proposed by L. A. Rubel, University of Illinois at Urbana-Champaign.*

(a) Let (z_n) and (z_n') be sequences in $\mathbb{C}$, neither with a finite limit point. Assume that if a complex number w occurs exactly k times in (z_n) where $k \geqslant 1$, then it occurs exactly $k-1$ times in (z_n'). (Subject to this restriction, we allow finite or even empty sequences.) Show that there exists an entire function f such that the zeros of f are exactly at the points of (z_n) and the zeros of f' are exactly at the points of (z_n'), with the proper multiplicity in each case.

(b) Can one similarly prescribe three sequences $(z_n), (z_n'), (z_n'')$ with corresponding assertions about the zeros of f, f', and f''?

6571. *Proposed by Glenn Ierley, Michigan Technological University, Houghton, MI.*

(a) Let $A(n)$ be the maximum area of a polygon with n sides of lengths $1, 2, \ldots, n$, where $n \geqslant 4$. It is known that the maximum area occurs for a polygon inscribed in a circle. (Cf. G. Pólya, *Mathematics and Plausible Reasoning*, Volume 1, Princeton, 1954, pp. 174–177.) Let $B(n)$ be the area of a regular polygon with n sides and perimeter $1 + 2 + \cdots + n$. Prove that

$$1 - \frac{A(n)}{B(n)} \sim \frac{\pi^2}{3n^2}. \qquad (n \to \infty).$$

(b) For $1/2 < q < 1$ let $A(q, n)$ be the maximum area of a polygon with n sides of lengths $1, q, q^2, \ldots, q^{n-1}$, respectively, where n is large enough so that $q + q^2 + \cdots + q^{n-1} > 1$. Let $B(q, n)$ be the area of a regular polygon with n sides and perimeter $1 + q + q^2 + \cdots + q^{n-1}$. Prove that $c(q) = \lim_{n\to\infty} A(q, n)/B(q, n)$ exists and find

$$\lim_{q \to 1^-} \frac{1 - c(q)}{(1-q)^2}.$$

SOLUTIONS OF ADVANCED PROBLEMS

Persistence of a Distribution Function

6522 [1986, 485]. *Proposed by Gunnar Blom, University of Lund and Lund Institute of Technology, Sweden.*

Let $X_1, X_2, \ldots$ be an infinite sequence of independent random variables with the common continuous distribution function F. Let X_N be the first variable that is less than exactly one of all its predecessors $X_1, \ldots, X_{N-1}$. Determine the distribution function of X_N.

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC, Canada. The distribution function of X_N is F. In fact, for any positive integer m this statement is true if "exactly one" is replaced by "exactly m".

For any $k > m$, the probability that X_k is less than exactly m of its predecessors is $1/k$ (since X_k is equally likely to be the first, second, ..., kth order statistic). Note that this is independent of the ordering of $X_1, \ldots, X_{k-1}$ among themselves. Thus the probability that X_k is the first one less than exactly m of its predecessors is

$$P(X_k = X_N) = \left(1 - \frac{1}{m+1}\right)\left(1 - \frac{1}{m+2}\right) \cdots \left(1 - \frac{1}{k-1}\right)\frac{1}{k} = \frac{m}{k(k-1)}.$$

Since the set of values of a given set of i.i.d. random variables is independent of the order in which they occur, the conditional distribution of X_N given $X_N = X_k$ is the distribution of the $(k-m)$th order statistic for $X_1, \ldots, X_k$. Namely, for any x, if $P(X_i \leq x) = p$, and $q = 1 - p$, then

$$P(X_N \leq x | X_N = X_k) = \sum_{j=0}^{m} \binom{k}{j} p^{k-j} q^j$$

so that

$$P(X_N \leq x) = \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \sum_{j=0}^{m} \binom{k}{j} p^{k-j} q^j.$$

Write this as R_m in order to make the dependence on m explicit. We shall prove by induction on m that $R_m = p$. Note that the series converges absolutely for $0 \leq p \leq 1$.

First consider the case $m = 1$ (which is the problem as stated). We have

$$\begin{aligned} R_1 &= \sum_{k=2}^{\infty} \frac{1}{k(k-1)} (p^k + kp^{k-1}q) \\ &= \sum_{k=2}^{\infty} \frac{1}{k(k-1)} (kp^{k-1} - (k-1)p^k) \\ &= \sum_{k=2}^{\infty} \left(\frac{p^{k-1}}{k-1} - \frac{p^k}{k} \right) = p. \end{aligned}$$

Now suppose that $R_{m-1} = p$. We have

$$R_m = \frac{m}{m-1} R_{m-1} - \frac{1}{m-1} \sum_{j=0}^{m-1} \binom{m}{j} p^{m-j} q^j + \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \binom{k}{m} p^{k-m} q^m,$$

where the sum over j comprises the terms for $k = m$ that are present in R_{m-1} but not in R_m, and the sum over k comprises the terms for $j = m$ that are present in R_m but not R_{m-1}. Now

$$\sum_{j=0}^{m-1} \binom{m}{j} p^{m-j} q^j = (p+q)^m - q^m = 1 - q^m$$

while

$$\begin{aligned} \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \binom{k}{m} p^{k-m} q^m &= q^m \sum_{i=1}^{\infty} \frac{(i+m-2)!}{i!(m-1)!} p^i \\ &= \frac{q^m}{m-1} (q^{1-m} - 1) = \frac{1}{m-1} (q - q^m). \end{aligned}$$

(Here we have used the binomial series

$$\sum_{i=0}^{\infty} \frac{(i+n)!}{i!n!} p^i = (1-p)^{-1-n},$$

which converges for $|p| < 1$; for $p = 1$ the formula is trivial.) Thus

$$R_m = \frac{m}{m-1}p - \frac{1}{m-1}(1 - q^m) + \frac{1}{m-1}(q - q^m) = p$$

as required.

The above generalization of 6522 was also proved by Barthel W. Huff, Eugene Salamin, and Glenn A. Stoops. Both Marcel F. Neuts and the proposer remark that it is sufficient to establish the result for the case in which F is the uniform distribution on $(0, 1)$. The proposer (who also provided a solution for $m = 1$ based on order statistics) used this to provide a noncalculational argument, based on the notion of "records," for the truth of the result. He adds that the result exists in the literature on "records" and is implicit in a paper of Charles M. Goldie and L. C. G. Rogers, The k-record Processes are i.i.d., *Z. für Wahrscheinlichkeitstheorie*, 67 (1984) 197–211.

Neuts added the following remark to his solution. "This result is quite remarkable. It shows that the distribution of the first near-record X_N is the same as that of the underlying random variables. This would be very difficult to infer, for example, from simulation runs. Because of the heavy tail of the distribution of N, the empirical distribution of X_N over many replicated runs may be expected to converge only very slowly to F."

Also solved by Thomas N. Delmer, Ellen Hertz, James M. Meehan, G. S. Rogers, Kenneth Schilling, David G. Weinman, Western Maryland College Problems Group, and Douglas P. Wiens (Canada).

Applied and Computational Complex Analysis. By Peter Henrici. Wiley-Interscience, 1986. vii 637 pp.

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This book is not a "textbook" in the ordinary sense of the word. The usual textbook is something that is trying to be all things to all people. The late Professor Henrici has written a book that is the distillation of his life's work as a scholar and research mathematician. The book is a lively and lovely piece of work and, like any truly interesting creation, it is full of tensions and contradictions.

The fact that there is a section on digital signal processing, in a book whose last chapter contains a proof of the Bieberbach Conjecture, may suggest that the author has made a rather eclectic choice of topics, but the flow of ideas is very orderly. The presentation of material is rigorous, but the author avoids overburdening the reader by choosing hypotheses for his results that are weak enough for many practical problems, yet strong enough to yield accessible proofs. This is strikingly demonstrated in his discussion of Cauchy integrals in section 14.1 and the results of Calderon discussed in the notes at the end of the section. There are many existential results, but care is taken to ensure effective computation methods. For instance, in Professor Henrici's discussion in § 14.6, entitled "Cauchy Integrals on Straight Line Segments," he arrives at Theorem 14.6a. This is immediately followed by the statement that "Theorem 14.6a does not express what from a numerical point of view may be its most significant aspect." Professor Henrici then reformulates Theorem 14.6a as Algorithm 14.6b, which provides a numerical method for carrying out a desired calculation.

In the Introduction, he states:

> Authors who primarily write for professional mathematicians may cultivate a style where a large number of facts are presented as concisely and economically as possible. However, the present work is not directed exclusively, and perhaps not even primarily, toward such mathematicians. A lifelong career in teaching this kind of reader has convinced me that, however great their appreciation for the logical coherence of the subject, their even greater concern is why they should be interested in it. Thus, time and again, I have allotted valuable space to the task of motivating what is ahead. Moreover, whenever facts or "theorems" are stated—and there are plenty of these—I have endeavored to find formulations that in their essence are intelligible also to readers who did not memorize all the preceding definitions. If I am accused of wordiness and of being, on occasion, repetitive, this is the price I must pay for attempting to reach a larger audience.

The final contradiction is that this book is, in my opinion, admirably suited for educating mathematicians.

An Outline of Set Theory. By James M. Henle. Problem Books in Mathematics, Springer-Verlag, New York, 1986, viii + 145 pp.

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That there is something wrong with the present teaching of mathematics is indisputable. Controversy arises when one tries to pinpoint exactly what is wrong, and it intensifies when blame is apportioned or panaceas are offered. However, there are points upon which agreement can be had—particularly if the points are sufficiently blunted and they are not fingers of blame.

What can we agree on? Mathematical education in its broadest form is a failure: the average educated person is unaware that mathematics is still alive, that it is not the case that all mathematical discoveries were made long ago. At the 1986 International Congress of Mathematicians, the winners of the Fields Medals and the Nevanlinna Prize were asked by reporters about the use of the computer in their work, never imagining that the computer is a businessman's rather than a mathematician's tool (even though they correctly perceived what some mathematicians have not, that the computer is mathematical in nature). This ignorance is a matter of miseducation. Partly it is caused by the ignorance of the teachers (high school mathematics teachers generally receive their degrees in education and learn less mathematics than mathematics majors, usually very little modern mathematics), and partly by the textbooks: it has often been remarked how textbooks in the sciences include some history, while college calculus books, generally the only mathematics texts to include any history, only have a few biographical remarks ranging in length from footnotes to half-page paragraphs.

The mathematical training of mathematicians and engineers could also be improved. Twenty years ago a course in the calculus supplied the equivalent of mathematical maturity, the appropriate prerequisite to upper division course work. Since then, the calculus textbooks have been diluted by the elimination of proofs, the labelling as optional of all conceptual material, and the encouragement of passivity on the part of the student by providing ready-made reviews with boxed formulae. Linear algebra is now supposed to provide the bridge between the calculus and higher mathematics, but I see linear algebra texts (as well as texts on differential equations) being produced by the same visionless authors with the same panoply of brain-softening "aids" to students—the cancer is spreading.

Progressing yet farther away from any point of common agreement, I will even say that there is something wrong with education in higher mathematics. The "Definition-Theorem-Proof" style of textbook writing is the most efficient means of transmitting a large quantity of information and it should not be lightly discarded. In a classroom, the dead facts can be brought to life by an informed instructor. But what happens if the instructor himself was textbook trained? He will not be able to bring the artifacts to life, because they are as much artifacts to him as they are to his students. And what will the textbooks such an instructor writes look like?

How do we breathe new life into the teaching of mathematics? There are two proposals crystalizing in the pedagogic vapours. First there is Harold Edwards' exhortation to "Read the Masters" and his guidebooks for those wishing to do so. At the other extreme is the philosophy of letting the students "discover" for themselves, a philosophy inherent in Springer-Verlag's new series Problem Books in Mathematics. Both proposals work, but I am not equally satisfied with their workings. Whereas I can sing lofty paeans to Edwards and his achievements, I can only caution against putting too much faith in problems courses and the readily misapplied Moore method: For, the life these latter breathe into the subject is not the real thing, but an artificial life like Frankenstein's monster and, however much sympathy the monster may evoke, it is yet a teratological creation lacking a soul. James Henle's *An Outline of Set Theory* exemplifies this poverty of spirit.

Consider the actual historical development of set theory. Working on the uniqueness of Fourier series, Georg Cantor obtained his results first for series convergent at all points, then for series convergent at all but finitely many exceptional points, and finally for series with infinitely many nicely distributed exceptional points. The introduction of ordinal numbers was necessary for a definition of "nicely distributed" and Cantor's research changed directions. For this he drew fire from Leopold Kronecker.

The seemingly tiny fact that the cardinality of the real plane is the same as that of the real line is not so tiny when one considers it in its historical context: to Cantor, one-dimensional and two-dimensional space were two distinct objects and, despite his attempt to prove their dissimilarity, he actually proved them to be virtually the same! In a letter to Richard Dedekind about his construction of a one-one correspondence between the unit interval and the unit square, Cantor wrote "I see it, but I don't believe it." Dedekind replied that dimensionality might still be a genuine concept: he didn't think the one-one correspondence could be continuous. This was in 1874; in 1890 Peano showed that the unit square was a continuous image of the unit interval. Perhaps the map could be one-one as well. It was not until 1909 that L. E. J. Brouwer proved the general invariance of dimension.

Set theory explicitly assumed a foundational role around the turn of the century when Gottlob Frege decided to base all of mathematics on it. Cantor's theories of ordinal and cardinal numbers subsumed the arithmetic of the natural numbers, and both Cantor and Dedekind had set-theoretic constructions of the real numbers from there. Unfortunately, Frege's axiomatic system was flawed—as Bertrand Russell noted. Avoiding the paradoxes was something that worried the philosophers, but it had little or no effect on mathematics or the development of set theory. Ernst Zermelo's axiomatization in 1908 accompanied his second proof of the Well-Ordering Theorem from the Axiom of Choice. Zermelo's first proof had met with much criticism and he supplied his new proof with an explicit list of axioms used so that it may be more convincing. About twenty years later, he offered the world the cumulative hierarchy of sets as an intuitive model of these axioms—actually, of an improved set of axioms resulting from some fine tuning by Abraham Fraenkel and Thoralf Skolem.

This minihistory of set theory falls a bit short of completeness, but, inadequate as it is, it does indicate the dynamic life of the subject and the excitement that could be conveyed to a student with an historically based text, or, at least, a scholarly written one. It hints at what the potential mathematicians among the students may experience in their own lifetimes: the unexpected discovery of something interesting, the incredible surprises and complete reversals of one's beliefs or intuitions, the timespan in the solution of problems, the subsumption of one programme by another, and the slow crystalization of concepts. The mention of Kronecker and the criticism of Zermelo's original proof also bring the human element into the story, both in its destructive and constructive aspects—two aspects that are ever at hand.

Now consider the static picture offered by the problems text. Henle has tried to go beyond the "Definition-Theorem-Problem" format by prefacing the work with a $3\frac{1}{2}$ page introduction including history, philosophy, a statement of intent, and discussions of his method and bias. He also sprinkles occasional two or three sentence remarks of a historical or philosophical nature throughout his book. The picture he paints can be glimpsed from a few quotes from his history:

> Mathematics is a living creature, growing as occasions demand and circumstances permit. Every now and then it must pause to organize and reflect on what it is and where it comes from. This happened in the sixth century B.C. when Euclid thought he had derived most of the mathematical results known at the time from five postulates. By the end of the nineteenth century, it was ready to happen again... In searching for underlying principles, mathematicians were led naturally to sets... [mention of Russell's paradox]... Over the decades following the discovery of such problems, a collection of principles or axioms was formed which appeared (and still appears) to avoid paradoxes. The system is called Zermelo-Fraenkel set theory or ZF after its originators, Ernst Zermelo and Abraham Fraenkel. In addition to occupying a strategic location in mathematics, ZF is studied for itself by a growing number of mathematicians. The father of modern set theory was Georg Cantor.

The story outlined (I have not omitted anything central) is wholly false, but that is hardly the issue—one of the first corollaries of the general floccinaucinihilipilification of scholarship in mathematics is the irrelevance of history. What is at issue is the one-dimensionality and simplicity (simplistic-icity?) of the artificial life created. Real life—healthy life—is rich in its complexity; artificial life—like Rabbi Loeb's golem or Henle's history—is unhealthy in its simplicity.

If one concomitant of the problem solver's perspective is the attitude that history doesn't matter—any story will do (but, the simpler the better)—another is the attitude that the results themselves are not important. Like the fisherman who doesn't eat fish, the mathematical problem solver is only out for the hunt. And, the Great White Hunter leading the safari selects the game suitable for his clients, who will see but a little of the jungle—or, perhaps, only the savannah and none of the jungle. This is exactly what happens in the problems course, as is illustrated by *An Outline of Set Theory*: nothing is taken very far. Dedekind's construction of the reals stops with the definition of $1/r$, thus almost but not quite proving the reals to be a field. The treatment of ordinal exponentiation is given in connexion with the Goodstein-Kirby-Paris Theorem, a recent "combinatorial independence result" closely tied to the Cantor Normal Form for ordinals less than ε_0. Neither ε_0 nor the normal form are mentioned. Cardinal arithmetic is omitted from the discussion of cardinals. Etc.

Other than to suggest that he be raked over the coals, or at least flogged, for the sentence about Euclid, I do not want to attack Prof. Henle for having written an unscholarly book. In this he has been thoroughly professional. I wish but to attack this highly professional perspective of mathematics as mere problem solving. Although some may attribute the lifelessness and shallowness of his book to his dubious goal of running a problem solving course for average students, I think the blame lies entirely in the too narrow perspective. *An Outline of Set Theory* is a vivid illustration of the need for something more: To instill some life into our mathematics texts, we also need some scholarship.

TELEGRAPHIC REVIEWS

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books and software submitted for review should be sent to Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T: Textbook	P: Professional Reading	1–4: Semesters
C: Computer Software	L: Undergraduate Library	**: Special Emphasis
S: Supplementary Reading	13: Grade Level	??: Questionable

Readers are advised that price information is subject to change, that computer software is often available also on other machines, and that hardware variations often cause software incompatibilities. Selected books and software packages receive a second, more extensive review in the MONTHLY.

Mathematics Appreciation, T(13: 1). *Mathematics in Daily Life: Making Decisions and Solving Problems.* Joanne Simpson Growney. McGraw-Hill, 1986, xxiii + 453 pp, $33.95. [ISBN: 0-07-025015-4] Covers the use of elementary mathematics, mostly arithmetic, in problem solving and decision making. Does not emphasize specific mathematical topics. Intended for use in college or university "general education" mathematics courses. Could also be of interest to elementary and secondary school teachers. Algebra is a helpful, but not necessary, prerequisite. Very easy to read. Includes many exercises varying in level of difficulty. RH

Education, P, L. *Implementation Handbook for the Comprehensive Mathematics Program.* Manford Byrd, Jr. (Board of Education of the City of Chicago, 1819 W. Pershing Rd., Chicago, IL 60609), 1987, (P). *Kindergarten-Grade 3,* viii + 154 pp; *Grades 4-6,* viii + 191 pp; *Grades 7-8,* viii + 138 pp. A detailed set of objectives and examples for each topic to be covered in each reporting period in grades K-8 in the Chicago public schools. Includes creative calculator examples, as well as geometry, data analysis, measurement, and applications in each grade. LAS

History, P, L.** *Reminiscences About a Great Physicist: Paul Adrien Maurice Dirac.* Ed: Behram N. Kursunoglu, Eugene P. Wigner. Cambridge U Pr, 1987, xviii + 297 pp, $49.50. [ISBN: 0-521-34013-6] Two dozen reflections by close friends of Dirac on both scientific and personal associations. Contributors include his widow Margit Dirac, her brother Eugene Wigner, P.A.M. Dirac himself (on the inadequacies of quantum field theory), and many world-famous physicists such as Harish-Chandra, Fred Hoyle, and Abdus Salam. LAS

History, S(17), P*. *The Historical Development of Quantum Theory, Volume 5: Erwin Schrödinger and the Rise of Wave Mechanics, Part 2: The Creation of Wave Mechanics; Early Response and Applications 1925-1926.* Jagdish Mehra, Helmut Rechenberg. Springer-Verlag, 1987, ix + 613 pp, $79.95. [ISBN: 0-387-96377-4] Wave mechanics began when Schrödinger showed in his fundamental papers of 1926 that the quantum states of the hydrogen atom could be represented as an eigenvalue problem for an appropriate operator. This volume reconstructs the scientific attitude of this revolutionary period. Schrödinger's own thoughts and motivation have always been obscure due to his lack of correspondence. The authors attempt to address this by presenting in detail the contributions of de Broglie, Einstein, and others along with Schrödinger's own work. (*Part 1,* TR, January 1988.) MR

Foundations, T(14: 1), S**, L*.** *Bridge to Abstract Mathematics: Mathematical Proof and Structures.* Ronald P. Morash. Math. Ser. Random House, 1987, xiii + 395 pp, $24. [ISBN: 0-394-35429-X] A text for a course which bridges the gap between beginning courses like calculus and more advanced courses in which students deal seriously with mathematical proofs. Topics include sets, logic, applications of logic, methods of mathematical proof, relations, functions, and the construction of number systems. Readable with lots of exercises of varying difficulty and solutions to selected exercises. CEC

Foundations, P, L. *Particles and Paradoxes: The Limits of Quantum Logic.* Peter Gibbins. Cambridge U Pr, 1987, xi + 181 pp, $34.50; $11.95 (P). [ISBN: 0-521-33498-5; 0-521-33691-0] An exposition and exposé of attempts to intepret the paradoxes of quantum mechanics (e.g., the twin slit experiment) via quantum logic. The author holds that quantum logic—indeed, any logic—is inadequate to the task of providing a foundation for quantum mechanics. "We are left with a mystery" of "just how odd the physical world must be." LAS

Foundations, T(17-18), S, P, L. *Varieties of Con-*

structive Mathematics. Douglas Bridges, Fred Richman. London Math. Soc. Lect. Note Ser., V. 97. Cambridge U Pr, 1987, x + 149 pp, $19.95 (P). [ISBN: 0-521-31802-5] An introduction to the spirit and practice of modern constructive mathematics, requiring a minimal background in philosophy and formal logic. The book introduces three varieties of constructive mathematics: Bishop's constructive mathematics, Brouwer's intuitionistic mathematics, and the constructive recursive mathematics of the Russian school of Markov. LCL

Combinatorics, T(17-18), P, L. *Combinatorial Geometries.* Ed: Neil White. Encycl. of Math. & Its Applic., V. 29. Cambridge U Pr, 1987, xii + 212 pp, $39.50. [ISBN: 0-521-33339-3] This is volume two in a three-volume series on matroid theory. Topics covered include coordinatizations, matching theory, transversed and simplicial matroids, the Möbius function and combinatorial optimization. Includes exercises. LC

Number Theory, T*(16-18: 1), S, P, L*. *A Course in Number Theory and Cryptography.* Neal Koblitz. Grad. Texts in Math., V. 114. Springer-Verlag, 1987, 208 pp, $34. [ISBN: 0-387-96576-9] A good introduction to the elementary number theory and algebra (e.g., finite fields) underlying public-key cryptosystems, including an exposition of the elliptic curve approach. Unfortunately, the cheap fuzzy printing is unpleasant to look at. Springer shouldn't let this happen in the GTM series. BC

Number Theory, T(14: 1), S, L*.** *Elementary Number Theory.* Charles Vanden Eynden. Math. Ser. Random House, 1987, xii + 266 pp, $27. [ISBN: 0-394-35359-5] A well-written introduction to the subject. Extensive exercise sets include many routine problems as well as a good collection of theoretical problems. Lots of historical material. Advanced topics include public key cryptography, quadratic residues, continued fractions, some diophantine equations, and Pell's equation. CEC

Linear Algebra, T(15-17: 1). *Applied Linear Algebra, Third Edition.* Ben Noble, James W. Daniel. Prentice-Hall, 1988, xvi + 521 pp. [ISBN: 0-13-041260-0] Major changes in this edition include greater emphasis on triangular forms, over 500 new exercises, and additional material on some topics previously discussed only in exercises (e.g., the Cayley-Hamilton and Perron-Frobenius theorems). Some examples and problems using computer software for numerical linear algebra have also been added. (*First Edition,* TR, October 1969; *Second Edition,* TR, November 1978.) AO

Linear Algebra, P. *Invariant Theory and Superalgebras.* Frank D. Grosshans, Gian-Carlo Rota, Joel A. Stein. CBMS Reg. Conf. Ser. in Math., No. 69. AMS, 1987, xxi + 80 pp, $16 (P). [ISBN: 0-8218-0719-6] A superalgebra is one part symmetric, one part exterior, and one part "divided powers." The authors prove an extension of the "standard basis theorem," and apply it to compute invariants of symmetric and skew-symmetric tensors. Includes a very clear and fairly long synopsis, and many examples. BC

Linear Algebra, T?(13: 1). *A First Course in Linear Algebra with Concurrent Examples.* A.G. Hamilton. Cambridge U Pr, 1987, viii + 148 pp, $39.50; $12.95 (P). [ISBN: 0-521-32516-1; 0-521-31041-5] Not enough material for a standard course. Covers solutions to linear equations, linear dependence, determinants through the 3x3 case, and some three-dimensional geometry. Answers to most exercises in back. GG

Linear Algebra, T*(14: 1, 2), L. *Linear Algebra.* John B. Fraleigh, Raymond A. Beauregard. Addison-Wesley, 1987, xv + 519 pp, $35.95. [ISBN: 0-201-15459-5] Includes standard topics: linear systems, vector spaces and linear transformations, determinants, eigenvalues, and orthogonality, all with emphasis on R^n. No Jordan forms. Applications include computer solutions of linear systems and eigenvalues, least squares, and linear programming. Also has chapter on calculus applications. Has answers to odd exercises; software to accompany text available. Good for course with applied flavor. GG

Linear Algebra, T(16-17: 1), S*, L. *Nonnegative Matrices and Applicable Topics in Linear Algebra.* Alexander Graham. Math. & Its Applic. Halsted Pr, 1987, 264 pp, $89.95. [ISBN: 0-470-20855-4] Introduction for general readers, includes background material (including aspects of graph theory; unitary, Hermitian, and normal matrices; positive definite matrices), followed by the main theme of the book and applications. Important concepts amply illustrated with worked examples and problems (with solutions). LCL

Group Theory, P. *Lecture Notes in Mathematics-1281: Group Theory.* Ed: O.H. Kegel, F. Menegazzo, G. Zacher. Springer-Verlag, 1987, vii + 179 pp, $16.30 (P). [ISBN: 0-387-18399-X] A collection of 19 short papers which were presented at the International Conference on Group Theory held at Brixen/Bressanone, Italy, May 25-31, 1986. The papers consider quite a range of topics; many involve nilpotent groups and several concern locally finite groups. LW

Group Theory, S(18), P. *Unitary Representations of Reductive Lie Groups.* David A. Vogan, Jr. Annals of Math. Stud., No. 118. Princeton U Pr, 1987, x + 308 pp, $60; $19.50 (P). [ISBN: 0-691-08481-5; 0-691-08482-3] General survey based loosely on January 1986 Hermann Weyl Lectures given at the Institute for Advanced Study. Introduction gives excellent perspective. Some proofs omitted or sketched. GG

Group Theory, P. *Essays in Group Theory.* Ed: S.M. Gersten. Math. Sci. Res. Inst., V. 8. Springer-Verlag, 1987, 342 pp, $32. [ISBN: 0-387-96618-8] A collection of five papers based on a seminar presented in 1985 at MSRI, Berkeley. Baumslag and Shalen examine presentations by the classical method of representations; Stalling and Gersten deal with applications of diagrams to group theory, and Gromov and Shalen with hyperbolic groups and actions on R-trees. The papers are all well-written and free of the typical terseness of papers appearing in journals. LW

Algebra, T(18: 1, 2), P. *Commutator Theory for Congruence Modular Varieties.* Ralph Freese, Ralph McKenzie. London Math. Soc. Lect. Note Ser., V. 125. Cambridge U Pr, 1987, 227 pp, $27.95 (P). [ISBN: 0-521-34832-3] A development of a theory of commutators in the setting of congruence modular varieties. This theory generalizes the notion of the commutator of a group. Includes many exercises to solutions and extensive bibliographical notes. SG

Algebra, P. *Noetherian Rings and Their Applications.* Ed: Lance W. Small. Math. Surv. & Mono., No. 24. AMS, 1987, ix + 118 pp, $38. [ISBN: 0-8218-1525-3] A collection of six papers (at just over $6 apiece) on such topics as the Goldie rank of a module, semisimple Lie algebras, and Noetherian group rings. SG

Algebra, P. *Lecture Notes in Mathematics-1271: Algebraic Groups, Utrecht 1986.* Ed: A.M. Cohen, *et al.* Springer-Verlag, 1987, viii + 284 pp, $24.30 (P). [ISBN: 0-387-18234-9] Fourteen papers from the April 1986 symposium honoring the 350th anniversary of the University of Utrecht and T.A. Springer's 60th birthday. GG

Algebra, T(14-15: 1). *Rings and Factorization.* David Sharpe. Cambridge U Pr, 1987, ix + 111 pp, $14.95 (P); $34.50. [ISBN: 0-521-33718-6; 0-521-33072-6] A very readable and enjoyable introduction to the concepts of rings, fields, prime elements, and unique factorization. Assumes no background in abstract algebra. Includes examples of concrete applications, such as factoring polynomials and Fermat's two-squares theorem. Contains many exercises along with hints and solutions. A fun and smooth introduction to some abstract mathematical ideas. RH

Algebra, P. *Structures Paragraduées (Groupes, Anneaux, Modules).* Marc Krasner, Mirjana Vuković. Papers in Pure & Appl. Math., No. 77. Queen's U, 1987, 163 pp, (P).

Algebra, T*(16: 1, 2), S, L*. *Introduction to Abstract Algebra.* Elbert A. Walker. Math. Ser. Random House, 1987, viii + 355 pp, $30. [ISBN: 0-394-35611-X] A sophisticated introduction to algebra for undergraduates. Chapter titles include sets, groups, vector spaces, rings and modules, linear transformations, fields, and topics from both group and ring theory. Includes excellent collections of exercises, but few of them are routine. Plenty of material for a year-long course. CEC

Algebra, P. *Lecture Notes in Mathematics-1280: Jordan Triple Systems by the Grid Approach.* Erhard Neher. Springer-Verlag, 1987, xii + 193 pp, $20 (P). [ISBN: 0-387-18362-0]

Real Analysis, P. *Regular Variation, Extensions and Tauberian Theorems.* J.L. Geluk, L. de Haan. CWI Tract, V. 40. Math Centrum, 1987, 132 pp, Dfl. 20.30 (P). [ISBN: 90-6196-324-9] A self-contained introduction to the theory of regular variation and its main extensions. It is shown that regularly varying functions are a natural setting for Tauberian theorems of the Laplace type. Also includes some results for general kernel transforms. Includes references. CEC

Complex Analysis, S(18), P. *Lectures on Counterexamples in Several Complex Variables.* John Erik Fornæss, Berit Stensønes. Math. Notes 33. Princeton U Pr, 1987, 247 pp, $22.50 (P). [ISBN: 0-691-08456-4] Lecture notes from a graduate course at Princeton. Begins with a brief introduction to basics of several complex variables. Topics include counterexamples to smoothing of plurisubharmonic functions, CR-manifolds, Stein neighborhood basis, peak sets, inner functions, Runge exhaustion. Presented in a clear, but somewhat informal style. No problems. BH

Complex Analysis, P. *Lecture Notes in Mathematics-1275, 1276, 1277: Complex Analysis.* Ed: Carlos A. Berenstein. Springer-Verlag, 1987, (P). *I,* xv + 331 pp, $28.60, [ISBN: 0-387-18356-6]; *II,* ix + 320 pp, $28.60, [ISBN: 0-387-18357-4]; *III,* x + 350 pp, $32.90. [ISBN: 0-387-18355-8] Proceedings of the sixteenth Special Year at the University of Maryland from July 1985 to December 1986. A collection of 54 papers on topics from complex analysis, including both survey articles and new results. RH

Complex Analysis, T*(16-17: 2, 3), L. *Complex Functions: An Algebraic and Geometric Viewpoint.* Gareth A. Jones, David Singerman. Cambridge U Pr, 1987, xiv + 342 pp, $59.50; $16.95 (P). [ISBN: 0-521-30893-3; 0-521-31366-X] Presents "main ideas about complex functions and Riemann surfaces" assuming only fundamentals of abstract algebra, real and complex analysis, and topology. Topics are Riemann sphere, Möbius transformations, elliptic functions, analytic continuation and Riemann surfaces, hyperbolic geometry, and the modular group. Well motivated with good selection of problems at end of each chapter. BH

Differential Equations, P. *Perturbation Methods, Bifurcation Theory and Computer Algebra.* Richard H. Rand, Dieter Armbruster. Appl. Math. Sci., V. 65. Springer-Verlag, 1987, ix + 243 pp, $29.80 (P). [ISBN: 0-387-96589-0] Contains MACSYMA programs for several popular perturbation methods as well as methods for studying bifurcations. A brief introduction to MACSYMA is provided as an appendix. AO

Differential Equations, P. *Ordinary and Partial Differential Equations.* Ed: B.D. Sleeman, R.J. Jarvis. Longman Scientific & Technical (US Distr: Wiley), 1987, 216 pp, $48.95 (P). [ISBN: 0-470-20839-2] Proceedings of the ninth Dundee Conference, 1986, with fourteen papers by seventeen authors; topics include coherent excitations, one- and two-parametric bifurcation diagrams, inverse scattering, limit cycle configurations of quadratic systems, concentration effects in Euler equations, homoclinic orbits and chaos in delay equations, the discrete Nagumo equation, radiation of sound by a moving sphere, fisheries stock assessment, Cahn-Hilliard model of phase transition, second-order elliptic equations, biHamiltonian systems, applications of differential equations in biology, and fibered structures in optimal design. RSF

Differential Equations, T*(16-17: 1, 2). *Applied Mathematics: A Contemporary Approach.* J. David Logan. Wiley, 1987, xviii + 572 pp, $44.95. [ISBN: 0-471-85083-7] A survey text covering most of the traditional topics and a few nontraditional ones as well: dimensional analysis and scaling, perturbation methods, calculus of variations, partial differential equations of mathematical physics, stability and bifurcation, similarity methods, and numerical techniques for partial differential equations. AO

Partial Differential Equations, P. *Contributions to Nonlinear Partial Differential Equations, Volume II.* Ed: J.I. Díaz, P.L. Lions. Res. Notes in Math. Ser., V. 155. Longman Scientific & Technical (US Distr: Wiley), 1987, 308 pp, $57.95 (P). [ISBN: 0-470-20810-4] The proceedings of the Second Franco-Spanish Colloquium on nonlinear partial differential equations held in Paris in December 1985. CEC

Partial Differential Equations, P. *Lecture Notes in Mathematics-1248: Nonlinear Semigroups, Partial Differential Equations and Attractors.* Ed: T.L. Gill, W.W. Zachary. Springer-Verlag, 1987, ix + 185 pp, $18 (P). [ISBN: 0-387-17741-8] This volume constitutes the proceedings of the symposium on nonlinear semigroups, partial differential equations, and attractors held at Howard University in Washington, D.C. on August 5-8, 1985. CEC

Partial Differential Equations, P. *Lecture Notes in Mathematics-1241: Singularities in Linear Wave Propagation.* Lars Gårding. Springer-Verlag, 1987, 125 pp, $13.10 (P). [ISBN: 0-387-18001-X] The aim of these lectures is to present the use of microlocal theory in the analysis of singularities in linear wave propagation. LCL

Partial Differential Equations, P. *Hyperbolic Equations.* Ed: F. Colombini, M.K.V. Murthy. Res. Notes in Math. Ser., V. 158. Longman Scientific & Technical (US Distr: Wiley), 1987, 286 pp, $51.95 (P). [ISBN: 0-470-20869-4] Proceedings of the conference on Hyperbolic Equations and Related Topics, University of Padova, 1985. LC

Partial Differential Equations, S(18), P. *Viscosity Solutions and Optimal Control.* Robert J. Elliott. Res. Notes in Math. Ser., V. 165. Longman Scientific & Technical (US Distr: Wiley), 1987, 95 pp, $39.95 (P). [ISBN: 0-470-20918-6] Discusses the concept of a viscosity solution and its relationship to a cost/value function of an optimal control problem or differential game. A viscosity function is any solution to the Hamilton-Jacobi-Bellman differential equation. Topics covered include dynamic programming and differential games with alternate, overlapping play. MR

Partial Differential Equations, P. *Wave Motion: Theory, Modelling, and Computation.* Ed: Alexandre J. Chorin, Andrew J. Majda. Math. Sci. Res. Instit., V. 7. Springer-Verlag, 1987, 336 pp, $32. [ISBN: 0-387-96594-7] Twelve papers from the proceedings of a conference given at the Mathematical Sciences Research Institute, Berkeley, California, June 1986, in honor of the 60th birthday of Peter Lax. LCL

Partial Differential Equations, P. *The Homotopy Index and Partial Differential Equations.* Krzysztof P. Rybakowski. Universitext. Springer-Verlag, 1987, xii + 208 pp, $39.50 (P). [ISBN: 0-387-18067-2] Homotopy index theory originated on compact spaces. As usual, the non-compact case is problematic. The author gives a unified approach, with applications to partial differential equations and other topics (e.g., Morse theory). BC

Numerical Analysis, P. *Lecture Notes in Engineering-27: The Best Approximation Method: An Introduction.* Th. V. Hromadka II, Ch.-Ch. Yen, G.F. Pinder. Springer-Verlag, 1987, xiii + 168 pp, $26.70 (P). [ISBN: 0-387-17572-5] A generalized Fourier expansion method for solving linear operator equations. Mathematical background in metric spaces and Lebesgue integration. Application to engineering problems including computer codes. RWN

Numerical Analysis, P. *Algorithms for Approximation.* Ed: J.C. Mason, M.G. Cox. Inst. of Math. & Its Applic. Conf. Ser., V. 10. Clarendon Pr, 1987, xvi + 694 pp, $125. [ISBN: 0-19-853612-7] 41 contributions at an international conference held July 15-19, 1985 at the Royal Military College of Science in Shrivenham, England. They concern algorithm development in spline approximation and smoothing, spline interpolation and shape preservation, multivariate interpolation, least-square methods, rational approximation, complex and nonlinear approximation, computer-aided design and blending; applications in numerical analysis, partial differential equations, other disciplines; software. DFA

Operator Theory, S(18), P. *Summing and Nuclear Norms in Banach Space Theory.* G.J.O. Jameson. Math. Soc. Stud. Texts, V. 8. Cambridge U Pr, 1987, xi + 174 pp, $39.50; $13.95 (P). [ISBN: 0-521-34134-5; 0-521-34937-0] Assumes basic knowledge of Banach and Hilbert space theory. Main topics are Pietsch's theorem on p-summing operators and Grothendieck's inequality and their applications. Exercises and numerous examples scattered throughout. BH

Functional Analysis, T(15: 1). *Introduction to the Analysis of Metric Spaces.* J.R. Giles. Australian Math. Soc. Lect. Ser., V. 3. Cambridge U Pr, 1987, xiv + 257 pp, $49.50; $16.95 (P). [ISBN: 0-521-35051-4; 0-521-35928-7] Follows educational style of Dieudonné used in British universities: detailed study of analysis of metric spaces before introducing topological spaces. Normed linear spaces treated as special subfamily of metric spaces. Very helpful geometrically-suggestive diagrams. Numerous good exercises at the end of each section. Connectedness, Axiom of Choice not included. MS

Functional Analysis, P. *Lecture Notes in Mathematics-1267: Geometrical Aspects of Functional*

Analysis. Ed: J. Lindenstrauss, V.D. Milman. Springer-Verlag, 1987, 212 pp, $20 (P). [ISBN: 0-387-18103-2] Proceedings of the Israel Seminar on geometric aspects of functional analysis held between October 1985 and June 1986. Central topic is convex sets in R^n and infinite-dimensional spaces. BH

Analysis, P. *Lecture Notes in Mathematics-1257: On the C^*-Algebras of Foliations in the Plane.* Xiaolu Wang. Springer-Verlag, 1987, 165 pp, $16.30 (P). [ISBN: 0-387-17903-8] A study of C^*-algebras of foliated open two manifolds, linking topology, graph theory, geometry, and analysis in an important new area of research. LCL

Analysis, P. *Lecture Notes in Mathematics-1265: Asymptotics for Orthogonal Polynomials.* Walter Van Assche. Springer-Verlag, 1987, vi + 201 pp, $20 (P). [ISBN: 0-387-18023-0] A monograph which concentrates on the asymptotic theory of general orthogonal polynomials on the real line. Includes a substantial list of references. CEC

Analysis, P. *A General Theory of Integration in Function Spaces, Including Wiener and Feynman Integration.* P. Muldowney. Res. Notes in Math. Ser., V. 153. Longman Scientific & Technical (US Distr: Wiley), 1986, 115 pp, $34.95 (P). [ISBN: 0-470-20736-1] Begins with a brief description of Henstock's general theory of integration (a simple, but powerful generalization of Riemann integration), then develops theory of function space integration in more detail and applies this theory to statistical mechanics (Wiener integral) and quantum mechanics (Feynman integral), thus giving a unified treatment of these two integrals. BH

Analysis, S(18), P, L. *Moments in Mathematics.* Ed: Henry J. Landau. Proc. of Symp. in Appl. Math., V. 37. AMS, 1987, xi + 154 pp, $30. [ISBN: 0-8218-0114-7] Applications and ramifications of the classical problem of moments (when does a sequence of numbers represent the successive moments of a non-negative measure?) to geometry, Hilbert space, signal processing, semigroups, probability and statistics. Papers from the January 1987 AMS short course in San Antonio. LAS

Analysis, P. *Value Distribution Theory for Meromorphic Maps.* Wilhelm Stoll. Aspects of Math., V. E7. Friedr. Vieweg & Sohn, 1985, xi + 347 pp, $28 (P). [ISBN: 3-528-08906-7] Generalizes standard Nevanlinna theory to the following setting: $f : M \to P_n$ is a meromorphic map, M is a parabolic manifold of dimension m, and P_n is n-dimensional projective space. Extensive bibliography. BH

Analysis, T(16-17). *Finite-Dimensional Spaces: Algebra, Geometry, and Analysis, Volume I.* Walter Noll. Mech.: Analy., V. 10. Martinus Nijhoff (US Distr: Kluwer Academic), 1987, xvi + 393 pp, $97.50. [ISBN: 90-247-3581-5] Based on a course originally entitled "Tensor Analysis" and then renamed "Multidimensional Algebra, Geometry, and Analysis," the text attempts to unify these topics and "to present mathematics as a whole." The prerequisites for a student are courses in abstract algebra, real analysis, and linear algebra. The topics covered include: dual, flat (affine), and inner-product spaces; basic point-set topology; basic differential calculus; spectral theory (concentrating on finite-dimensional spaces); and the structure of linear transformations. Each of the ten chapters is followed by a problem set. LW

Algebraic Geometry, S(18), P. *Géométrie algébrique réelle.* J. Bochnak, M. Coste, M-F. Roy. Ser. of Mod. Surv. in Math., Band 12. Springer-Verlag, 1987, x + 373 pp, $118.30. [ISBN: 0-387-16951-2] If you want to make algebraic geometry *really* complex, do it over R, not C. BC

Differential Geometry, T(18: 1), S*, P*. *Teichmüller Theory and Quadratic Differentials.* Frederick P. Gardiner. Wiley, 1987, xvii + 236 pp, $46.95. [ISBN: 0-471-84539-6] Treats the moduli space of inequivalent Riemann surface structures on a given topological surface. Full proofs and exercises make the book self-contained for a reader with knowledge of a few topics beyond the basics of complex variables, differential geometry, and algebraic topology. GG

Geometry, S, L.** *A Budget of Trisections.* Underwood Dudley. Springer-Verlag, 1987, xv + 169 pp, $29.80. [ISBN: 0-387-96568-8] A witty, compassionate, and richly personal account of the countless amateur mathematical Don Quixotes who pursue their dream of trisecting angles. Echoing De Morgan's 1872 *A Budget of Paradoxes,* Dudley describes the trisectors and their trisections, showing that despite their ignorance of advanced mathematics, trisectors are often solid citizens who through clever persistence produce some very good approximations. LAS

Geometry, T(17-18: 2, 3), S, P. *Algorithms in Combinatorial Geometry.* Herbert Edelsbrunner. EATCS Mono. on Theor. Comput. Sci., V. 10. Springer-Verlag, 1987, xv + 423 pp, $49. [ISBN: 0-387-13722-X] Despite its title, this book offers an extensive introduction to combinatorial geometry, generally, and to its correlative field, computational geometry, before treating geometric algorithms and their applications. Exercises, research problems, and up-to-date bibliography. SS

Geometry, P. *Intuitive Geometry.* Ed: K. Bőröczky, G. Fejes Tóth. Colloquia Mathematica Societatis Janos Bolyai, V. 48. Elsevier Science, 1987, 708 pp, Dfl. 295. [ISBN: 0-444-879-33-1] Assortment of research papers on convexity, discrete and combinatorial geometries, and relations between geometry and other fields: biology, architecture, art, and psychology. SS

Algebraic Topology, P. *Lecture Notes in Mathematics-1274: Equivariant K-Theory and Freeness of Group Actions on C^*-Algebras.* N. Christopher Phillips. Springer-Verlag, 1987, viii + 371 pp, $32.90 (P). [ISBN: 0-387-18277-2] Given an action of a compact Lie group on a compact Hausdorff space, an examination of the equivariant K-theory reveals whether this group action is free. The category of compact Hausdorff spaces is (contravariantly) equivalent to the category of commutative unital C^*-algebras, so it becomes natural to try to extend the above result to general C^*-algebras. There being no completely satisfactory notion of freeness of an action on a C^*-algebra, the author herein defines and studies conditions on the equivariant K-theory of a C^*-algebra, and in case the algebra is commutative, is able to determine through the equivariant K-theory whether the underlying space is free. LW

Differential Topology, P. *Lecture Notes in Mathematics-1264: Rational Homotopy Theory.* Wu Wen-tsün. Springer-Verlag, 1987, viii + 219 pp, $20 (P). [ISBN: 0-387-13611-8] Clarifying and extending the works of D. Sullivan, this book studies the rational homotopy type (minimal model) of a differential graded algebra of differential forms on a simplicial complex K. In particular, the book shows that the minimal model is homotopically invariant, and contains complete information on the cohomology and homotopy ring of K. The author takes a constructional approach, and is thus able to explicitly determine the minimal model of a fiber space as well as that of a fiber-square-constructed space. LW

Topology, T*(17-18), S*, P, L. *Classical Tessellations and Three-Manifolds.* José Maria Montesinos. Universitext. Springer-Verlag, 1987, xvii + 230 pp, $35 (P). [ISBN: 0-387-15291-1] Written to provide graduate students with a source of geometrical intuition in low-dimensional topology, this fascinating monograph illuminates three of Thurston's eight geometries by exploring manifolds of tessellations in Euclidean, spherical, and hyperbolic space. Profusely illustrated with mineral crystal patterns, ornamental motifs, and mathematical sketching. LAS

Dynamical Systems, T(16-17), P, L. *Chaotic Vibrations: An Introduction for Applied Scientists and Engineers.* Francis C. Moon. Wiley, 1987, xv + 309 pp, $39.95. [ISBN: 0-471-85685-1] An applied view of chaotic phenomena, richly illustrated with physical systems, mathematical models, and standard examples (Henon maps, Lorenz attractors, Poincaré maps, Lyapunov exponents). Little direct mathematical theory, but a great deal of illustrative applications that help to develop intuition. LAS

Dynamical Systems, P. *Attraktoreingrenzung für nichtlineare Systeme.* Gennadij A. Leonov, Volker Reitmann. Teubner-Texte zur Math., V. 97. BG Teubner, 1987, 196 pp, 20M (P). [ISBN: 3-322-00427-9]

Dynamical Systems, P. *Autowave Processes in Kinetic Systems: Spatial and Temporal Self-Organization in Physics, Chemistry, Biology, and Medicine.* V.A. Vasiliev, *et al.* Math. & Its Applic. D Reidel (US Distr: Kluwer Academic), 1987, 262 pp, $69. [ISBN: 90-277-2379-6] From Chapter 1: "An autowave process is a self-sustained wave process in an active nonlinear medium maintaining its characteristics at a constant level at the expense of an energy source distributed in the medium." The key word is "nonlinear." Examples include propagation of phase-transition fronts, Belousov-Zhabotinskii reactions, and nerve cell activity. BC

Probability, S, L**.** *Fifty Challenging Problems in Probability with Solutions.* Frederick Mosteller. Dover, 1987, viii + 88 pp, $3.95 (P). [ISBN: 0-486-65355-2] Unabridged republication of the work first published in 1965. Stimulating and challenging problems with leisurely, instructive, conversational solutions that demonstrate mathematical thinking at its best. LCL

Probability, P*. *Dependence in Probability and Statistics: A Survey of Recent Results.* Ed: Ernst Eberlein, Murad S. Taqqu. Prog. in Prob. & Stat., V. 11. Birkhauser Boston, 1986, xi + 473 pp, $39.50. [ISBN: 0-8176-3323-5] Twenty-three papers written by participants in an international conference held in Oberwolfach in April 1985, concerned with limiting results assuming various dependence structures. "These papers tie together known results, describe the underlying ideas, summarize the state-of-the-art, and state some open problems." RSK

Stochastic Processes, P*. *Stochastic Geometry and Its Applications.* D. Stoyan, W.S. Kendall, J. Mecke. Prob. & Math. Stat. Wiley, 1987, 345 pp, $49.95. [ISBN: 0-471-90519-4] Revised translation of Stoyan and Mecke's 1983 book *Stochastische Geometrie.* "Topics covered include the basic theories of point processes, random sets, fibre processes, tessellations, stereology and the statistical theory of shape." Extensive bibliography of this relatively new field. RSK

Stochastic Processes, S(17-18), P. *Radically Elementary Probability Theory.* Edward Nelson. Annals of Math. Stud., No. 117. Princeton U Pr, 1987, ix + 97 pp, $40; $15 (P). [ISBN: 0-691-08473-4; 0-691-08474-2] Develops the theory of probability and stochastic processes using nonstandard analysis, leaving the results in nonstandard form. More advanced appendix shows how the conventional theory of stochastic processes can be derived from nonstandard theory. Note hardcover price. RSK

Stochastic Processes, T(18: 1, 2). *Applied Probability and Queues.* Søren Asmussen. Prob. & Math. Stat. Wiley, 1987, x + 318 pp, $67.95. [ISBN: 0-471-91173-9] Divided into three parts: Part A deals with simple Markovian models, particularly as they relate to queueing theory; Part B presents basic mathematical tools—renewal theory, regenerative processes, and random walks; Part C investigates special models and methods, primarily dealing "with a more narrow class of problems associated with general distributions of interarrival times and service times." RSK

Elementary Statistics, T*(13: 1), S. *Introduction to Biostatistics, Second Edition.* Robert R. Sokal, F. James Rohlf. WH Freeman, 1987, xii + 363 pp, $37.95. [ISBN: 0-716-71805-7] Modest revision of the authors' 1973 *First Edition* (TR, March 1974), again based on their more inclusive book *Biometry (Second Edition,* TR, May 1982). Changes include some new descriptive procedures, more nonparametrics, the Bonferroni method of multiple comparisons, and the use of the G statistic rather than the more common chi-square test to analyze frequency data. RSK

Elementary Statistics, T(14-16), S, L. *Elements of Statistics for the Life and Social Sciences.* Braxton M. Alfred. Texts in Stat. Springer-Verlag, 1987, xiii + 190 pp, $36. [ISBN: 0-387-96500-9] Expansive, self-contained introduction to statistical ideas needed for scientific work in anthropology (and presumably other life and social sciences as well). Case studies used to illustrate logical argument, deductive thinking, prediction, and hypothesis testing. No exercises. LCL

Elementary Statistics, T(14-15: 1, 2). *Basic Statistical Methods for Engineers and Scientists, Third Edition.* John B. Kennedy, Adam M. Neville. Harper & Row, 1986, xx + 613 pp. [ISBN: 0-06-043633-6] Revision of the authors' 1976 *Second Edition.* Includes new material on probability, statistical inference, and goodness of fit. Emphasizing practical considerations, it also includes chapters on rejection of outliers, distributions of extremes, tolerance and control charts, acceptance and rejection testing, and an introduction to the design of experiments. RSK

Statistics, P. *Advances in Multivariate Statistical Analysis: Pillai Memorial Volume.* Ed: A.K. Gupta. Theory & Dec. Lib., Ser. B. D Reidel (US Distr: Kluwer Academic), 1987, xvi + 389 pp, $89. [ISBN: 90-277-2531-4] Twenty-one papers providing a cross-section of recent developments in multivariate statistical analysis. Dedicated to the memory of K.C. Sreedharan Pillai (1920-1985), it contains a short biography and a bibliography of his works. Note price. RSK

Statistics, T(17-18: 1), S, P*. *The Asymptotic Theory of Extreme Order Statistics, Second Edition.* Janos Galambos. Robert E Krieger, 1987, xv + 414 pp, $49.50. [ISBN: 0-89874-957-3] Revision of the author's 1978 *First Edition* published by Wiley (TR, December 1978). Rigorous presentation covering all known asymptotic models. Extensive bibliography, supplemented by a survey of the literature in each chapter. RSK

Statistics, T(17), P, L. *Kendall's Advanced Theory of Statistics, Fifth Edition of Volume 1: Distribution Theory.* Alan Stuart, J. Keith Ord. Oxford U Pr, 1987, xvi + 604 pp, $75. [ISBN: 0-19-520561-8] Major revision of the 1977 *Fourth Edition* of the first volume of this classic three-volume treatise, originally written by Kendall (1907-1983). The topics covered remain basically the same, but much new material has been added and terminology and proofs have been updated. RSK

Statistics, T(14-16: 1, 2), S, L. *Applied Statistics: Analysis of Variance and Regression, Second Edition.* Olive Jean Dunn, Virginia A. Clark. Prob. & Math. Stat. Wiley, 1987, xii + 445 pp, $36.95. [ISBN: 0-471-81269-2] Revision of the authors' 1974 text (TR, February 1975). Each chapter now includes a brief description of relevant BMDP and SAS computer programs. Also includes new material on repeated measure designs and the use of dummy variables in multiple regression and covariance analysis, and expanded material on variable selection and testing assumptions in multiple regression. RSK

Statistics, T(18: 1), P. *Asymptotic Distribution Theory in Nonparametric Statistics.* Manfred Denker. Adv. Lect. in Math. Friedr Vieweg & Sohn, 1985, vii + 204 pp, $18 (P). [ISBN: 3-528-08905-9] Treats three basic types of statistics: Hoeffding's U-statistics, differentiable statistical functionals, and statistics based on ranks. Concludes with a chapter on contiguity and efficiency. No exercises. RSK

Statistics, T(17: 1, 2), P. *Plane Answers to Complex Questions: The Theory of Linear Models.* Ronald Christensen. Texts in Stat. Springer-Verlag, 1987, xiv + 380 pp, $42. [ISBN: 0-387-96487-8] Designed "to rigorously illustrate the practical application of the projective approach to linear models." First half covers standard topics in regression analysis, analysis of variance and covariance; last part introduces various special topics such as residual analysis, variance component estimation, and log-linear models. RSK

Statistics, T(18: 1), S, P*. *Measurement Error Models.* Wayne A. Fuller. Prob. & Math. Stat. Wiley, 1987, xxiii + 440 pp, $44.95. [ISBN: 0-471-86187-1] Presents both theory and applications of models in which the explanatory variables are measured with error. Includes a number of real examples illustrating the procedures. Good bibliography. RSK

Statistics, S(16-17), P. *Design, Data, and Analysis by Some Friends of Cuthbert Daniel.* Ed: Colin L. Mallows. Prob. & Math. Stat. Wiley, 1987, xix + 380 pp, $29.95. [ISBN: 0-471-83937-X] Seventeen papers, three expository, two concerned with design issues, and the other twelve dealing primarily with the analysis of data, written in recognition of Daniel's great influence in this latter area. Written for "students and aspiring statistical consultants." Includes a brief outline of Daniel's career and a list of his publications. RSK

Statistics, P. *Contributions to the Theory and Application of Statistics: A Volume in Honor of Herbert Solomon.* Ed: Alan E. Gelfand. Academic Pr, 1987, xxviii + 544 pp, $59.95. [ISBN: 0-12-279450-8] Twenty papers, contributed by friends and colleagues, grouped into four areas where Solomon has made significant contributions: operations research and applied probability; distribution theory and geometric probability; applications in the areas of law and justice, medicine and psychology; and inference methodology. Includes a brief biographical sketch and a list of Solomon's publications. RSK

Statistics, T(16-17: 1), S, L. *Introduction to Statistical Inference.* Jack Carl Kiefer. Ed: Gary Lorden. Texts in Stat. Springer-Verlag, 1987, viii + 334 pp, $48. [ISBN: 0-387-96420-7] Text based on lecture notes developed by Kiefer (1924-1981) for a first course in statistical inference. Presents a modern decision-theoretic approach to inference, emphasizing the need to justify the use of a procedure by some criterion of goodness. RSK

Statistics, P. *The Population-Sample Decomposition Method.* A.M. Wesselman. Intern. Stud. in Econ. & Econometrics, V. 19. Kluwer Academic, 1987, 242 pp, $47.50. [ISBN: 90-247-3603-X] An approach which does not require all the model assumptions of classical statistical methods. The study aims to illustrate how the population-sample decomposition method is applicable to a much wider class of statistics. LCL

Programming, T(13-14: 1). *Pascal-SC: A Computer Language for Scientific Computation.* Gerd Bohlender, *et al.* Perspect. in Comput., V. 17. Academic Pr, 1987, ix + 292 pp, $34. [ISBN: 0-12-111155-5] Pascal-SC (scientific computation—implemented for Z80, 8088, and 68000 processors) is an extension of standard Pascal which allows greater accuracy in solving numerical problems, and has dynamic arrays and better string handling; chapters (among others) include standard Pascal, real floating-point arithmetic, strings and text processing, dynamic arrays, and modules; a syntax diagram appendix is included, along with numerous complete examples; useful as either a reference or text for a numerical-based Pascal course. RSF

Languages, P. *Lecture Notes in Computer Science-274: Functional Programming Languages and Computer Architecture.* Ed: Gilles Kahn. Springer-Verlag, 1987, vi + 470 pp, $34.60 (P). [ISBN: 0-387-18317-5] 24 papers from the third conference on the title subject, held in Portland, Oregon, September 1987. This conference focused particularly on implementation techniques for functional programming languages and on computer architectures to support the efficient execution of functional programs. DFA

Languages, P. *REDUCE: Software for Algebraic Computation.* Gerhard Rayna. Symbolic Computation. Springer-Verlag, 1987, ix + 329 pp, $29.80 (P). [ISBN: 0-387-96598-X] An introduction to REDUCE 3—a system for symbolic algebraic computation available on systems ranging in scale from an IBM PC to a Cray X/MP. Of special interest is a section containing several case studies illustrating the use of this symbolic computation package to solve nontrivial problems. AO

Languages, P. *Lecture Notes in Computer Science-276: ECOOP '87: European Conference on Object-Oriented Programming.* Ed: J. Bézivin, *et al.* Springer-Verlag, 1987, vi + 273 pp, $23.10 (P). [ISBN: 0-387-18353-1] Contains 25 papers on various aspects of object-oriented programming. AO

Computer Systems, P. *Lecture Notes in Computer Science-275: System Development and Ada.* Ed: A.N. Habermann, U. Montanari. Springer-Verlag, 1987, 305 pp, $25.70 (P). [ISBN: 0-387-18341-8] An interesting collection of papers presented at the workshop on software factories and Ada held in Capri, Italy, May 1986. Covers three general topics. First, design of software development environments, describing commercially available environments to ease production of very large software systems. Language-specific and mixed-language environments are presented. Secondly, methods for software development, addressing the issue of ensuring that implementations accurately reflect specifications. Finally, Ada compiler validation and specifying and testing Ada events are covered. PS

Theory of Computation, P. *Lecture Notes in Computer Science-273: A Connotational Theory of Program Structure.* James S. Royer. Springer-Verlag, 1987, 186 pp, $18 (P). [ISBN: 0-387-18253-5] This monograph is an outgrowth of a Ph.D. dissertation (SUNY at Buffalo) continuing development of a language-independent theory of program structure begun by Riccardi and Case; the central theme is the subclass, called acceptable numberings, of effective numberings; also stressed is "building" one control structure from others. It is very mathematical, with symbolism, numerous definitions, lemmas, propositions, and theorems; includes a bibliography, notation index, and definition index. RSF

Theory of Computation, P. *Lecture Notes in Computer Science-270: Computation Theory and Logic.* Ed: Egon Börger. Springer-Verlag, 1987, ix + 442 pp, $34.60 (P). [ISBN: 0-387-18170-9] Memorial volume for Dieter Rödding, a German logician interested in interaction of logic and computer science. Topics include recursion theory, automata theory, complexity theory, applications of logic in systems theory. KS

Theory of Computation, P. *Logic of Programming and Calculi of Discrete Design.* Ed: Manfred Broy. NATO ASI Ser. F, V. 36. Springer-Verlag, 1987, 413 pp, $75. [ISBN: 0-387-18003-6] Proceedings of a NATO Advanced Study Institute held in Marktoberdorf, Federal Republic of Germany, July 29-August 10, 1986. Contains fourteen papers on methodologies for the specification, design, and verification of programs. AO

Theory of Computation, T*(16-18: 1, 2), S, P, L. *Foundations of Logic Programming, Second, Extended Edition.* J.W. Lloyd. Symbolic Computation. Springer-Verlag, 1987, xii + 212 pp, $44.50. [ISBN: 0-387-18199-7] Expanded material in the *Second Edition* leans toward databases. As well, the class of programs considered is enlarged. Preliminaries are in the first chapter while the second chapter contains the declarative and procedural semantics of definite programs. Chapter three deals with normal programs and chapter four is new and treats unrestricted programs. The last chapter provides a theoretical foundation for deductive database systems. Chapter problems. References. Index. (*First Edition,* TR, October 1985.) RJA

Artificial Intelligence, P. *Natural Language Parsing Systems.* Ed: Leonard Bolc. Symbolic Computation. Springer-Verlag, 1987, xviii + 367 pp, $49.50. [ISBN: 0-387-17537-7] This collection of nine essays (fourteen authors) about current problems of natural language parsing in an artificial intelligence context presents research results in greater detail than journal articles allow; each paper has an abstract and reference list; there is a volume subject index. RSF

Artificial Intelligence, P. *The Knowledge Frontier: Essays in the Representation of Knowledge.* Ed: Nick Cercone, Gordon McCalla. Symbolic Computation. Springer-Verlag, 1987, xxxv + 512 pp, $42. [ISBN: 0-387-96557-2] This collection of 17 essays (26 authors) about knowledge representation is an outgrowth of the IEEE Computer Special Issue on Knowledge Representation in 1983; six of the papers are new, 7 are updated, and 4 are unchanged; the papers are organized in six sections—Overview, Logic, Foundations, Organization, Reasoning, and Applications, giving a comprehensive treatment to this aspect of artificial intelligence. The preface, index, and reference list are extensive. RSF

Computer Science, P. *Lecture Notes in Computer Science-263: Advances in Cryptology—CRYPTO '86.* Ed: A.M. Odlyzko. Springer-Verlag, 1987, xi + 487 pp, $34.60 (P). [ISBN: 0-387-18047-8] Proceedings of an annual conference held at the University of California, Santa Barbara, devoted to cryptologic research. Thirty-six papers classified by data encryption, public-key cryptography, zero-knowledge proofs, secret-sharing methods, hardware systems, software systems, software protection, probabilistic methods, and other topics. LCL

Computer Science, P. *Lecture Notes in Computer Science-265: Analogical and Inductive Inference.* Ed: K.P. Jantke. Springer-Verlag, 1987, vi + 227 pp, $20.60 (P). [ISBN: 0-387-18081-8] Written versions of presentations given at the International Workshop on Analogical and Inductive Inference held in Wendisch-Rietz, GDR, October 6-10, 1986. Discussion centered around two basic approaches to learning algorithms: inductive inference and analogical reasoning. RJA

Computer Science, P. *Lecture Notes in Computer Science-264: Logic Programming '86.* Ed: Eiiti Wada. Springer-Verlag, 1987, 179 pp, $18 (P). [ISBN: 0-387-18024-9] Proceedings of the Fifth Logic Programming Conference held in Tokyo, Japan, June 23-26, 1986. RJA

Applications, P. *Reliability Data Bases.* Ed: Aniello Amendola, Alfred Z. Keller. D Reidel (US Distr: Kluwer Academic), 1987, xii + 398 pp, $72. [ISBN: 90-277-2549-7] Probabilistic techniques are increasingly used to assess safety, new system designs, and reliability of products. These applications require the availability of good dependable and relevant data about failure rates, incidents, performance, and so forth. These proceedings (ISPRA Course, October 1985) provide a comprehensive review of the state-of-the-art relating to the collection, processing, and use of such data. LCL

Applications, P. *Filtering for Stochastic Processes with Applications to Guidance, Second Edition.* Richard S. Bucy, Peter D. Joseph. Chelsea, 1987, xviii + 217 pp, $19.95. [ISBN: 0-8284-0326-0] Filtering a random signal process from observations corrupted by noise. A reprint of a 1968 edition which gives a detailed derivation of the Kalman-Bucy filter (TR, May 1969). Includes appendices which correct the *a priori* bounds and gives an up-to-date list of relevant references. CEC

Applications (Biological Science), P. *Mathematical Aspects of Hodgkin-Huxley Neural Theory.* Jane Cronin. Stud. in Math. Biology, V. 7. Cambridge U Pr, 1987, xi + 261 pp, $49.50. [ISBN: 0-521-33482-9] An introduction to the study of mathematical models of electrically active cells. Provides an account of the literature on the subject and covers in detail the Hodgkin-Huxley model for nerve conduction. Accessible to mathematicians with little or no background in physiology. RH

Applications (Cognitive Science), S, P, L.** *Brains, Machines, and Mathematics, Second Edition.* Michael A. Arbib. Springer-Verlag, 1987, xvi + 202 pp, $27. [ISBN: 0-387-96539-4] Revision (principally an expanded introduction) of a path-breaking book written in 1962 (published by McGraw-Hill in 1964) outlining a theory of automata to explain the processing of information in brain-like machines. Beginning with the McCulloch and Pitts model of neural networks and ending in Gödel's theorem, Arbib joins biology to mathematics in a prescient challenge to contemporary cognitive science. LAS

Applications (Electrical Engineering), T(15-16: 1). *Integrated Circuit Design.* Alan F. Murray, H. Martin Reekie. Springer-Verlag, 1987, xiv + 152 pp, $28. [ISBN: 0-387-91303-3] Designed as a textbook for an introductory course for electrical engineering or physics students. The final chapter provides a design exercise that can be assigned as a year-long project. AO

Applications (Electrical Engineering), T(16-17: 2), S, P, L. *Methods in Electromagnetic Wave Propagation.* D.S. Jones. Oxford Eng. Sci. Ser., V. 8. Clarendon Pr, 1987, $29.95 each (P). *Volume 1: Theory and Guided Waves,* 433 pp, [ISBN: 0-19-856189-X]; *Volume 2: Radiating Waves,* 476 pp. [ISBN: 0-19-856190-3] The objective of these volumes is to develop a suitable framework of theory and numerical analysis with applications to various aspects of propagation of electromagnetic waves. Numerous exercises are included. MU

Applications (Engineering), P. *Reliability Modelling and Applications.* Ed: A.G. Colombo, A.Z. Keller. D Reidel (US Distr: Kluwer Academic), 1987, x + 391 pp, $74. [ISBN: 90-277-2566-7] Proceedings of the ISPRA Course (November 1985) includes twenty papers grouped into five sections: systems reliability, availability and maintainability of systems, structural reliability, reliability of computing systems, human reliability. LCL

Applications (Engineering), P. *Lecture Notes in Mathematics-1279: Saint-Venant's Problem.* Dorin Ieşan. Springer-Verlag, 1987, viii + 162 pp, $16.30 (P). [ISBN: 0-387-18361-2] The problem of A.J.C.B. de Saint-Venant is to "determine the equilibrium of a homogeneous and isotropic linearly elastic cylinder, loaded by surface forces distributed over its plane ends." These lecture notes present recent research on Saint-Venant's problem for anisotropic elastic bodies, on the relaxed Saint-Venant problem for heterogeneous elastic cylinders, and on Saint-Venant problems within the linearized theory of Cosserat elastic bodies. One hundred sixty references from Saint-Venant's original papers to the present. JK

Applications (Engineering), P. *Contact Mechanics.* K.L. Johnson. Cambridge U Pr, 1987, xi + 452 pp, $34.50 (P). [ISBN: 0-521-34796-3] Covers stresses and deformations which arise when two solid bodies come in contact, especially when the contact is localized due to dissimilar profiles (non-conformal). This induces a local stress concentration which can be studied using the method of superposition of point force solutions. MR

Applications (Engineering), P. *Lecture Notes in Engineering-9: The Complex Variable Boundary Element Method.* Theodore V. Hromadka II. Springer-Verlag, 1984, xi + 243 pp, $22 (P). [ISBN: 0-387-13743-2] Develops methods for approximating solutions to the Dirichlet and Neumann boundary-value problems by applying the Cauchy integral formula to the boundary element method. Includes a quick review of basic complex variable theory. BH

Applications (Fluid Mechanics), P. *Amorphous Polymers and Non-Newtonian Fluids.* Ed: Constantine Dafermos, J.L. Ericksen, David Kinderlehrer. Inst. for Math. & Its Applic., V. 6. Springer-Verlag, 1987, xii + 195 pp, $22. [ISBN: 0-387-96556-4] Ten papers from a workshop held at the Institute for Mathematics and Its Applications (IMA) in Minnesota during the 1984-85 special year on continuum physics and partial differential equations. LCL

Applications (Physics), P, L. *Three Hundred Years of Gravitation.* Ed: S.W. Hawking, W. Israel. Cambridge U Pr, 1987, xiii + 684 pp, $69.50. [ISBN: 0-521-34312-7] Sixteen chapters, separately authored, surveying cosmology, gravitation, string theory, and other theories descended from Newton's *Principia,* published just 300 years ago. Much more than a simple collection of papers, this comprehensive commemorative volume is a superb, well-planned exposition of the modern theory of gravitation. LAS

Applications (Physics), T(16-17: 1, 2), S, L. *Gravitational Physics of Stellar and Galactic Systems.* William C. Saslaw. Cambridge U Pr, 1987, xvii + 491 pp, $34.50 (P). [ISBN: 0-521-34975-3] Although primarily a theoretical treatise based on classical Newtonian gravity, some observational data is introduced for motivation. The text is relatively self-contained and accessible to anyone familiar with advanced calculus. Numerous interesting exercises are included. MU

Applications (Physics), P. *Symmetries of Maxwell's Equations.* W.I. Fushchich, A.G. Nikitin. Transl: John R. Schulenberger. Math. & Its Applic. D Reidel (US Distr: Kluwer Academic), 1987, xiv + 214 pp, $74. [ISBN: 90-277-2320-6] The power of Maxwell's equations lies largely in their symmetries, both geometric (Lie-group) and non-geometric (Lie-algebra). The former is classical; the latter is still being developed, and is the main subject here. The moral: Symmetry makes the world go 'round. BC

Applications (Simulation), P. *DEMOS: A System for Discrete Event Modelling on Simula.* G.M. Birtwistle. Springer-Verlag, 1987, 215 pp, $18 (P). [ISBN: 0-387-91301-7] An introduction to discrete event simulation modelling using DEMOS—a system which compliments SIMULA by providing elements which help beginners write simulations more quickly. Describes the basic DEMOS approach to model building as well as DEMOS descriptions of synchronization problems arising in discrete event simulations. Written as a teaching text for DEMOS, not as a reference. Tutorial style presents new features and offers an illustrative example of each. DEMOS programs' advantage over regular SIMULA programs: more easily written and understood without a thorough knowledge of SIMULA. Includes exercises and solutions. PS

Applications (Social Science), S, P, L. *Multidimensional Similarity Structure Analysis.* I. Borg, J. Lingoes. Springer-Verlag, 1987, xiv + 390 pp, $39. [ISBN: 0-387-96525-4] A class of models that represent similarity coefficients among a set of objects (e.g., correlation matrix) as distances in multidimensional space (two points are closer together when they are more correlated). The resulting picture is easier to assimilate than the table of coefficients. This book of case studies deals with all aspects of this subject, starting from scratch assuming no more than a high school background in mathematics. LCL

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