THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 71 NUMBER 6

CONTENTS

JUNE-JULY 1964

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

BIRKHOFF'S AXIOMS FOR SPACE GEOMETRY

ROLAND BROSSARD, Université de Montréal

1. Introduction. The motivation for this study is a definition of euclidean geometry leaving open the possibility of extension to higher dimensional spaces, based on the intuitive ideas concerning the use of graduated rulers and protractors. In fact, the only essential change in the Birkhoff's presentation is a weakening in the protractor axiom; this weakening allows the geometry of three-dimensional space to be constructible.

The system of axioms is based on "coordinate functions"; they are intuitively conceived as "applications" of a long graduated ruler to the lines, and as "applications" of a protractor to the plane bundles of half-lines. Distance, angular measure, and the betweenness relation are defined in terms of coordinate functions. The axioms are formulated in such a way as to exhibit a certain symmetry between the properties of coordinate functions for the elements of the lines and the properties of coordinate functions for the elements of the bundles, the main difference consisting in the value-rings.

2. Primitive notions. *Points* are abstract undefined objects. Primitive terms are: point, line, coordinate function of a line, half-line, bundle of half-lines, and coordinate function of a bundle of half-lines.

3. Axioms on points, lines, and coordinates of the points of the lines. Certain subclasses of points are called *lines*. The axioms on the lines are:

L_1. *There exist at least two distinct points.*

L_2. *If A and B are two distinct points, then there exists one and only one line containing A and B.*

L_3. *There exist points not all on the same line.*

A set of points is said to be *collinear* if this set is a subset of a line. Two sets are *collinear* if the union of these two sets is collinear. Coordinate functions on the lines are introduced by the following axiom.

CL_1. *There exists associated with each line* L, *a nonempty class X of one-to-one mappings x of L onto the field R of real numbers. If x_i is a member of X and if x_j is any one-to-one mapping of* L *onto R, then x_j is a member of X if and only if for all $A \in$*L *and for all $B \in$*L.

$$|x_i(A) - x_i(B)| = |x_j(A) - x_j(B)|.$$

The elements of X are called *coordinate functions* of L. The *distance* between two points A and B, denoted AB (or BA) is defined to be the unique nonnegative number $|x(A) - x(B)|$ where x is an arbitrary member of X. The point B is *between* the points A and C if A, B, and C belong to the same line and either $x(A) < x(B) < x(C)$ or else $x(C) < x(B) < x(A)$. We shall now show that this betweenness relation is defined independently of the coordinate function considered on the line containing the points A, B, and C.

If B is between A and C with respect to a coordinate function x_i then

(3.1) $x_i(A) < x_i(B) < x_i(C)$ or else (3.2) $x_i(C) < x_i(B) < x_i(A)$.

Let x_j be an arbitrary coordinate function for the same line. Axiom CL_1 implies that

$$x_i(A) - x_i(B) = x_j(A) - x_j(B) \quad \text{or else} \tag{3.3}$$

$$x_i(A) - x_i(B) = x_j(B) - x_j(A), \tag{3.4}$$

$$x_i(A) - x_i(C) = x_j(A) - x_j(C) \quad \text{or else} \tag{3.5}$$

$$x_i(A) - x_i(C) = x_j(C) - x_j(A), \tag{3.6}$$

$$x_i(B) - x_i(C) = x_j(B) - x_j(C) \quad \text{or else} \tag{3.7}$$

$$x_i(B) - x_i(C) = x_j(C) - x_j(B). \tag{3.8}$$

If (3.1) is valid then all left members of (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) are negative and if (3.2) is valid the same left members are all positive. Equations (3.3), (3.5), and (3.7) are valid or else equations (3.4), (3.6), and (3.8) are valid because any two equations in a group implies the third one. Consequently with respect to x_j, B is also between A and C.

For collinear points A, B, C the point B is between the points A and C if and only if $AB+BC=AC$. If O and A are two distinct points of a line, we call *half-line* OA with *end-point* O the set of points P on that line such that O is not between A and P. In speaking of a half-line OA, the first element of the ordered couple (O, A) will always represent the end-point. If A and B are distinct points, the set of points containing A, B, and all the points between A and B is called a *segment*. The distance AB is called the *length* of the segment AB. A segment without its end-points is an *interval*.

M is the *mid-point* of the segment AB if M is an element of this segment and if $AM=MB$. If x is a coordinate function for the line AB then there exists on that line a point M defined by the relation

$$x(M) = \frac{x(A) + x(B)}{2}.$$

We can easily verify that M is between A and B (i.e., M belongs to the segment AB), and that $AM=MB$. Consequently every segment has a mid-point. There is only one mid-point because if M and M' are two mid-points of the segment AB, then $A\neq B$, M and M' belong to the line AB, $AM=MB$, $AM'=M'B$ and there exists a coordinate function x for the line AB such that

$$x(A) - x(M) = x(M) - x(B)$$

and

$$x(A) - x(M') = x(M') - x(B);$$

consequently $x(M')-x(M)=x(M)-x(M')$, $x(M)=x(M')$, and $M=M'$.

4. Axioms on bundles and coordinates of the half-lines of the bundles. Certain subclasses of the class of all half-lines with the same end-point are called *bundles*. The common endpoint O of the elements of a bundle is called the *vertex* of the bundle; the notation B_o will be used for a bundle of vertex O. An *angle* is an unordered couple of half-lines with the same end-point O; the point O is called the *vertex* of the angle, and the half-lines the *sides* of the angle. An angle is *straight* if the sides are distinct and collinear. The axiom on the bundle is:

B_1. *If l and m are two noncollinear half-lines with the same end-point O, then there exists one and only one bundle B_o containing these half-lines.*

The axiom on the coordinate functions of the bundles is:

CB_1. *There exists, associated with each bundle B_o, a nonempty class Φ of one-to-one mappings ϕ of B_o onto the equivalence classes of real numbers modulo 2π. If ϕ_i is a member of Φ and if ϕ_j is any one-to-one mapping of B_o onto the equivalence classes of real numbers modulo 2π, then ϕ_j is a member of Φ if and only if for all $l \in B_o$ and for all $m \in B_o$*

$$|\phi_i(l) - \phi_i(m)| \equiv |\phi_j(l) - \phi_j(m)|,$$

where $|\phi_i(l) - \phi_i(m)| \equiv |\phi_j(l) - \phi_j(m)|$ *stands for*

$$\phi_i(l) - \phi_i(m) = (\phi_j(l) - \phi_j(m)) \pmod{2\pi}$$

or

$$\phi_i(l) - \phi_i(m) = (\phi_j(m) - \phi_j(l)) \pmod{2\pi}.$$

The elements of Φ are called *coordinate functions* of B_o. If x denotes a real number, we shall denote by $[x]$ the equivalence class modulo 2π containing x, and we shall denote by $\bar{y}$ the real number of the class $[y]$ such that $0 \leqq \bar{y} < 2\pi$. And $x \equiv y$ will mean $x = y$ (modulo 2π).

Let l, m be an angle belonging to a bundle B_o; the *measure* of the angle l, m with respect to B_o, denoted $\angle lm$, is the minimum of the two real numbers $\overline{\phi(l) - \phi(m)}$, $\overline{\phi(m) - \phi(l)}$ where ϕ is a coordinate function associated to the bundle B_o. $\angle AOB$ is independent of the coordinate function used to obtain it. If A, O, B are three distinct points, we write $\angle AOB$ for $\angle$(half-line OA)(half-line OB); the measure being calculated in a bundle B_o containing the half-lines OA and OB. We shall prove in the next section that the measure of an angle is independent of the bundle B_o in which this angle is embedded.

5. The continuity axiom and the similarity axiom.

CONTINUITY AXIOM. *If B_o is a bundle of vertex O, and if A, B are distinct nonvertex points of noncollinear half-lines of the bundle, then to every point P on the segment AB, there exists a half-line OC of B_o containing P such that*

$$[\angle AOP + \angle POB] = [\angle AOB].$$

Conversely if a half-line OC of the bundle B_o is such that $[\angle AOC + \angle COB] = [\angle AOB]$ *then there exists a point P belonging simultaneously to the halfline OC and to the segment AB.*

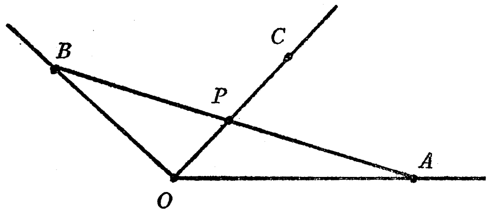

FIG. 1

For an angle defined by two noncollinear half-lines OA and OB, axiom B_1 implies that there exists only one bundle B_o containing this angle; therefore $\angle AOB$ is uniquely defined. If OA and OB coincide, the class of bundles containing the angle AOB may have more than one element, but in that case the measure of this angle is zero whatever the considered bundle of the class. We shall now show that the measure of a straight angle is π whatever the bundle in which this straight angle is embedded.

If an angle AOP has π for measure then the half-line OA is distinct from the half-line OP (otherwise $\angle AOP = 0$). If this angle AOP with measure π is not a straight angle then there exists a point B on the line AP, not on the line OA, such that P is between A and B (Fig. 1), and in the unique bundle B_o containing the half-lines OA, OB, OP the continuity axiom implies that $[\angle AOP + \angle POB] = [\angle AOB]$, that is to say,

$$[\pi] + [\angle POB] = [\angle AOB]. \tag{5.1}$$

But $\angle POB \neq 0$ for the points A, O, P are not collinear. Furthermore $\angle POB \neq \pi$, for if $\angle POB = \pi$ then from (5.1) we have $\angle AOB \equiv \pi + \angle POB \equiv \pi + \pi \equiv 2\pi \equiv 0$ and OA coincides with OB. Consequently (5.1) is contradictory and if an angle, with respect to a bundle containing this angle, has π for measure then this angle is a straight angle.

Let AOB be a straight angle and let B_o be an arbitrary bundle containing this angle. Then for an arbitrary admissible coordinate function ϕ for B_o we have either (i) $0 \leqq \overline{\phi(OA)} < \pi$ or else (ii) $\pi \leqq \overline{\phi(OA)} < 2\pi$. In case (i) let OC be the unique element of B_o such that $\phi(OC) = \phi(OA) + [\pi]$; then $\phi(OC) - \phi(OA) = \phi(OA) + [\pi] - \phi(OA) = [\pi]$, the angle AOC having, with respect to B_o, π for measure is a straight angle, and consequently the angle AOB, being equal to the angle AOC, also has π for measure. Similarly, in case (ii) let OD be the unique element of B_o such that $\phi(OD) = \phi(OA) - [\pi]$; then $\phi(OA) - \phi(OD) = \phi(OA) - (\phi(OA) - [\pi]) = [\pi]$, the angle AOD is straight, and the angle AOB has also π for measure.

This completes the proof that the measure of an angle is independent of the bundle in which it can be contained. Furthermore we have proved the following theorem concerning the measure of straight angles.

THEOREM 1. *The measure of an angle is π if and only if this angle is straight.*

We shall now consider properties of the bundles and their coordinate functions.

THEOREM 2. *If $\phi\in\Phi$ and if ϕ_1 is any mapping of B_o into the equivalence classes of real numbers modulo 2π, then $\phi_1\in\Phi$ if and only if there exist $\epsilon=[\pm 1]$, and θ such that for every $l\in B_o$, $\phi_1(l)=\epsilon\phi(l)+\theta$.*

Proof. Sufficiency is obvious. To prove necessity suppose that $\phi_1\in\Phi$. Because ϕ is one-to-one there exist half-lines m, n in B_o such that $\phi(m)=[0]$, $\phi(n)=[\pi]$. Let $\phi_1(m)=\theta$. Then

$$\phi_1(n) - \phi_1(m) = \pm (\phi(n) - \phi(m)) = \pm [\pi] = [\pi],$$

and so

$$\phi_1(n) = [\pi] + \theta. \tag{5.2}$$

For any $l\in B_o$ we have

$$\phi_1(l) - \phi_1(m) = \pm (\phi(l) - \phi(m)),$$

so

$$\phi_1(l) = \epsilon_l\phi(l) + \theta, \qquad \epsilon_l = [\pm 1].$$

Let $S=\{l\mid \epsilon_l=1\}$, $T=\{l\mid \epsilon_l=-1\}$. We obviously have $m\in S\cap T$, and from (5.2), $n\in S\cap T$. Let $l'\in S$ and $l''\in T$. Then $\phi_1(l')-\phi_1(l'')=\phi(l')+\theta+\phi(l'')-\theta=\phi(l')+\phi(l'')$. But we also have either

$$\phi_1(l') - \phi_1(l'') = \phi(l') - \phi(l'')$$

or

$$\phi_1(l') - \phi_1(l'') = - \phi(l') + \phi(l'').$$

In the first case we get $2\phi(l'')=[0]$, $\phi(l'')=[0]$ or $[\pi]$, $l''=m$ or n, and $l''\in S$; in the second case $l'\in T$. Hence, either S or T contains all the elements of B_o and ϵ_l is uniformly $[1]$ or $[-1]$. This proves the theorem.

The intuitive content is clear. Two coordinate functions for a bundle are related in such a way that one can be deduced from the other by a rotation of the protractor (graduated from 0 to 2π) or an inversion of the protractor followed by a rotation. The corresponding proposition for coordinate functions on the lines is also true; the proof is almost the same.

COROLLARY 1. *If m and n are two noncollinear half-lines of a bundle B_o, then there exists one and only one coordinate function ϕ such that $\overline{\phi(m)}=0$ and $\overline{\phi(n)}<\pi$.*

Proof. Let $\phi_1 \in \Phi$. A necessary and sufficient condition for ϕ is that for some ϵ and θ

$$\phi(l) = \epsilon\phi_1(l) + \theta$$

with

$$(5.3) \qquad \begin{aligned} [0] &= \epsilon\phi_1(m) + \theta, \\ \phi(n) &= \epsilon\phi_1(n) + \theta. \end{aligned}$$

Eliminating θ gives

$$\phi(n) = \epsilon(\phi_1(n) - \phi_1(m)).$$

Since m and n are noncollinear, $\phi_1(n) - \phi_1(m) \neq [0]$ or $[\pi]$, and hence there is a unique choice of $\epsilon = [\pm 1]$ such that $\phi_1(n) < \pi$. From (5.3), θ is then also uniquely determined.

We can here observe that the measurement of angles can be conceived intuitively as being obtained in the usual way, that is to say, application of a protractor (plain or half-disk) in such a way that one side of the angle coincides with the zero of the protractor and such that the other side corresponds to a number less than π (the measure of the angle).

COROLLARY 2. *If l, m, n are distinct elements of a bundle, and if l and n are collinear, then* $\angle lm + \angle mn = \pi$.

For if ϕ is the unique coordinate function such that $\overline{\phi(l)} = 0$ and $\overline{\phi(m)} < \pi$, then $\overline{\phi(n)} = \pi$ and $\angle lm + \angle mn = \pi$.

COROLLARY 3. *If l is a half-line of a bundle B_o and if α is a positive real number less than π and different from zero, then there exist two and only two distinct half-lines m, n such that* $\angle lm = \angle ln = \alpha$.

Proof. If ϕ is a coordinate function for B_o such that $\phi(l) = 0$ then there exist half-lines m and n such that $\phi(m) = -\phi(n) = [\alpha]$. Half-lines m and n have the required properties.

A *triangle* is an unordered set of three distinct points. The points are the *vertices* of the triangle. The three segments defined by the vertices of a triangle are the *sides* of the triangle. The three angles defined by the sides of a triangle are the *angles* of the triangle. In the context of triangles, for instance a triangle ABC, the measure of an angle, say angle ABC with vertex B, will be denoted $\angle B$ instead of $\angle ABC$. Two triangles are *similar* if the vertices can be labelled A, B, C and A', B', C' in such a way that

$$\text{(a)} \qquad AB/A'B' = BC/B'C' = CA/C'A',$$

$$\text{(b)} \qquad \angle A = \angle A', \qquad \angle B = \angle B', \qquad \angle C = \angle C'.$$

The constant ratio in (a) is called *factor of proportionality*. A triangle is *proper* if the vertices are noncollinear.

SIMILARITY AXIOM. *If two triangles ABC and $A'B'C'$ are such that $AB/A'B' = BC/B'C'$ and $\angle B = \angle B'$, then they are similar.*

6. Theorems on triangles.

THEOREM 4. *If two proper triangles ABC and $A'B'C'$ are such that $\angle A = \angle A'$ and $\angle B = \angle B'$, then they are similar.*

THEOREM 5. *If ABC is a proper triangle, then the sides AB and AC have equal length if and only if $\angle B = \angle C$.*

The proof of Theorem 4 and Theorem 5 can be found in [1].

LEMMA 1. *Let l be an element of a bundle B_o and let α, β be two numbers between zero and π. If m is a half-line of B_o such that $\angle lm = \alpha$, then there exists a half-line n of B_o with the following properties:* (a) $\angle ln = \beta$, (b) *for all points $A \in m$ and for all points $B \in n$ such that $A \neq B$ the segment AB has a point P in common with the unique line containing the half-line l.*

Proof. Let ϕ be a coordinate function such that $\phi(l) = [0]$ and $\phi(m) = [\alpha]$. Let n be a member of B_o such that $\phi(n) = [-\beta]$.

Case 1: $\alpha + \beta < \pi$. We have

$$(\phi(m) - \phi(l)) + (\phi(l) - \phi(n)) = \phi(m) - \phi(n),$$

i.e. $[\alpha] + (-([-\beta])) = [\alpha] - [-\beta]$, and $\angle lm + \angle ln = \angle mn$.

Case 2: $\alpha + \beta > \pi$. Let $\bar{l} \neq l$ be the half-line of B_o collinear with l; $\phi(\bar{l}) = [\pi]$. Then

$$(\phi(\bar{l}) - \phi(m)) + (\phi(n) - \phi(\bar{l})) = (\phi(n) - \phi(m)),$$

i.e. $([\pi] - [\alpha]) + (([-\beta]) - [\pi]) = [-\beta - \alpha] = [2\pi - (\alpha + \beta)]$, $[2\pi - (\alpha + \beta)] < \pi$, and $\angle \bar{l}m + \angle \bar{l}n = \angle mn$.

In both cases, the continuity axiom implies the desired result.

Case 3: If $\alpha + \beta = \pi$, then Theorem 1 implies that O is between A and B.

THEOREM 6. *If the triangles ABC and $A'B'C'$ are such that $AB/A'B' = BC/B'C' = CA/C'A'$, then they are similar.*

Proof. Case 1: A, B, and C are not collinear. Let $B_{A'}$ be a bundle of vertex A' containing the half-lines $A'B'$ and $A'C'$. Let l be a member of $B_{A'}$ such that (a) $\angle l(A'C') = \angle BAC$, (b) the segment $B'B''$, where B'' is on l and $A'B'' = A'B'$, has a point P in common with the line $A'C'$ (the existence of l is a consequence of Lemma 1). The triangles ABC and $A'B''C'$ are similar (similarity axiom). Consequently

$$AB/A'B'' = BC/B''C' = CA/C'A' = k = CA/C'A' = BC/B'C' = AB/A'B',$$

and $C'B' = C'B''$. The triangles $B'A'B''$ and $B'C'B''$ are isosceles and by Theorem 5 $\angle A'B'P = \angle A'B''P$, $\angle C'B'P = \angle C'B''P$.

If the collinear points A', P, C' are distinct, one of them is between the two others, and the continuity axiom implies that $\angle A'B'C' = \angle A'B''C'$ (e.g. if P is between A' and C' then

$$[\angle A'B'P + \angle PB'C] = [\angle A'B'C'] \quad \text{and} \quad [\angle A'B''P + \angle PB''C] = [\angle A'B''C'],$$

so that $\angle A'B'C' = \angle A'B''C'$). If P coincides with A' or with B' the same is obviously true. This proves that the triangles ABC and $A'B'C'$ are similar. We observe that if A, B, and C are not collinear, then A', B', and C' are also not collinear.

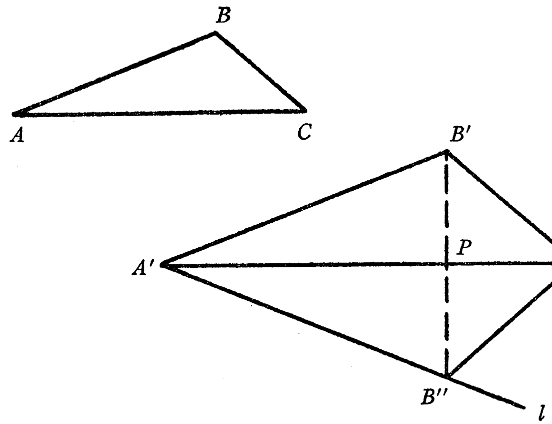

FIG. 2

Case 2: A, B, and C are collinear. In this case, A', B', and C' are also collinear. As A, B, and C are distinct, one of them, say B, is between the two others. It is sufficient to prove that the corresponding point B' is between A' and C'. Remembering that for collinear points A', B', C', the point B' is between the points A' and C' if and only if $A'B' + B'C' = C'A'$, we can see by the relations

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AB + BC}{A'B' + B'C'} = \frac{AC}{A'C'}$$

that if B' were not between A' and C' we would have a contradiction.

THEOREM 7. *The sum of the measures of the angles of a triangle is equal to* π.

Proof. We consider first the case where the triangle is a proper triangle with vertices A, B, and C. Let A', B', C' be the midpoints of the segments BC, CA, and AB respectively.

By multiple applications of the continuity axiom, we obtain:

1) A' between B and C implies the existence of a point E on line AA' and on line BB' between B and B';

2) B' between A and C implies the existence of a point E' on line BB' and on line AA' between A and A'; ABC being a proper triangle, $E = E'$;

3) A' between B and C implies the existence of a point D on line AA' and on line $C'B'$ between C' and B';

4) C' between A and B implies the existence of a point D' on line AE and on line $C'B'$ between A and E; again ABC being a proper triangle, $D=D'$. Furthermore E between A and A' and D between A and E imply that D is between A and A'. Then

$$[\angle AB'C' + \angle A'B'C'] = [\angle AB'A']$$

i.e., $\angle AB'C' + \angle A'B'C' = \angle AB'C'$ (for $\angle AB'C' < \pi$, and $\angle A'B'C' < \pi$), but $\angle AB'A' + \angle A'B'C = \pi$ and consequently $\angle AB'C' + \angle A'B'C' + \angle A'B'C = \pi$. The triangles ABC, $AB'C'$, $A'BC'$, and $A'B'C'$ being similar,

$$\angle AB'C' = \angle C, \qquad \angle A'B'C' = \angle B,$$
$$\angle A'B'C = \angle A, \qquad \angle A + \angle B + \angle C = \pi.$$

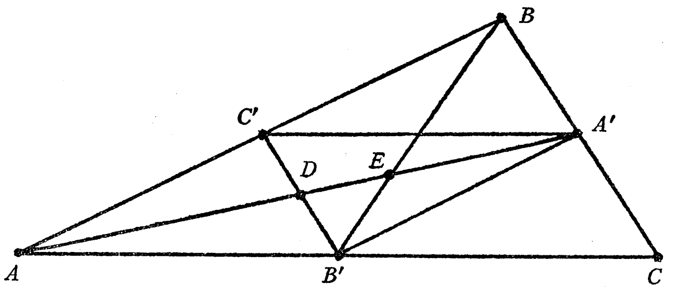

FIG. 3

To complete the proof, we observe that if the triangle is not proper, then one angle has for measure π and the two others have for measure zero.

As in [1] *congruence* of triangles is defined by similarity together with a factor of proportionality k equal to one.

Two distinct lines having a point in common determine six angles with nonnull measures. Two have π for measure and we can easily show that the four remaining ones form two sets, each set consisting of two distinct angles with the same measure. Two distinct lines having a point in common are said to be *perpendicular* if the four angles with measures different from zero and different from π have the same measure i.e., $\pi/2$.

LEMMA 2. *If L is a line and P is a point not on L, then there exists one and only one line containing P and perpendicular to L.*

Proof. Let A be an arbitrary point of L. There exists one and only one bundle of vertex A containing the lines L and AP. We shall denote this bundle by B_A. Let l' be the unique half-line of B_A, distinct from AP, such that $\angle ll' = \angle l(AP)$ where l is one of the half-lines with end-point A, determined by the line L and the point A. There exists on l' a point P' such that $AP'=AP$. Lemma 1 implies that the segment PP' has a point I in common with the line L.

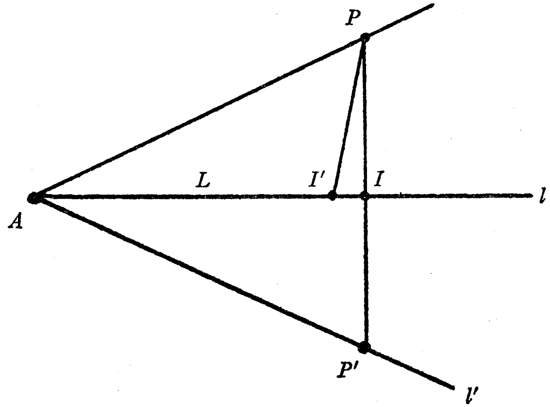

FIG. 4

If $I\neq A$, then the proper triangles API and $AP'I$ are congruent and $\angle AIP = \angle AIP' = \pi/2$ and PP' is perpendicular to L. If $I=A$, then PP' is also perpendicular to L. Furthermore, this perpendicular is unique because if there were to exist another perpendicular PI' to L the sum of the measures of the angles of the triangle PII' would be greater than π.

A triangle is *right-angled* if one of its angles has measure $\pi/2$.

LEMMA 3. *If a triangle ABC is right-angled at A, then the unique perpendicular from A to the line defined by the vertices B and C meets this line in a point D between B and C.*

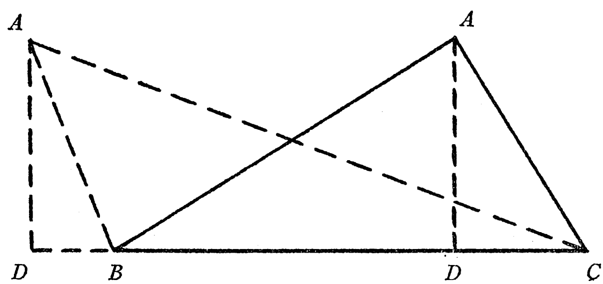

FIG. 5

Proof. If D coincides with B or with C, then the triangle would have two angles with measure $\pi/2$. If B were between D and C, we would have $[\angle DAB+\pi/2]=[\angle DAC]$, so that $\angle DAC>\pi/2$; the right-angled triangle DAC has then for sum of measures of its angles a number greater than π, which is a contradiction. In the same way, C cannot be between B and D, and D is between B and C.

THEOREM 8. *If a triangle ABC is such that $\angle A=\pi/2$, then $(AB)^2+(AC)^2=(BC)^2$.*

Proof. Let AD be the unique perpendicular from A to BC (Fig. 5). The triangles CAD and CBA are similar (Theorem 4), and

$$CD/AC = AC/BC. \tag{6.1}$$

In the same way the triangles ABD and CBA are similar, and as D is between B and C, we have

$$(BC - CD)/AB = AB/BC. \tag{6.2}$$

The elimination of CD between (6.1) and (6.2) gives the desired relation.

COROLLARY 3. *If A, B, and C are distinct points, then*

$$AB + BC \geqq AC. \tag{6.3}$$

The equality holds if and only if the points A, B, C are collinear with B between A and C.

Proof. If A, B, and C are collinear, then $AB+BC\geqq AC$. The equality holds if and only if B is between A and C. If A, B, and C are not collinear, let B' be the point of intersection of the unique perpendicular from B to AC with AC. Then Theorem 8 implies $AB>AB'$, $BC>B'C$, and $AB+BC>AB'+B'C\geqq AC$ (B' being between A and C or not). If A, B, C were not collinear and if $AB+BC=AC$, then we would have a contradiction.

COROLLARY 4. *Distance on the set of all points is a metric on this set, that is to say, $AB=BA$, $AB+BC\geqq AC$, $AB\geqq 0$, and $AB=0$ if and only if $A=B$.*

7. Parallel lines, and the concept of plane. A *plane* is defined to be the class of all points belonging to the half-lines of a bundle B_o; this class will be denoted by $\{B_o\}$.

THEOREM 9. *If two distinct points of a line are in a plane, then the whole line is in the plane.*

Proof. We know by the continuity axiom that if two distinct points P and B belong to a plane (Fig. 1), the points of the segment PB belong to the plane. If A is a point of the line PB not on the segment PB, then P is between A and B or B is between A and P; let P be between A and B. Then the half-lines OA and OB, where O is the vertex of the bundle B_o defining the plane, belong to a unique bundle $\overline{B}_o$, and the continuity axiom implies that the half-line OP is a member of $\overline{B}_o$. Then $\overline{B}_o=B_o$, and the line AB is in the plane $\{B_o\}$.

We shall now prove that a plane is uniquely determined by three noncollinear points. The proof will be preceded by two lemmas.

LEMMA 4. *If two bundles B_o and $B_{o'}$ are such that $O \neq O'$, if there exists a point A such that A is not on the line OO', and if $A, O, O' \in \{B_o\} \cap \{B_{o'}\}$, then the planes $\{B_o\}$ and $\{B_{o'}\}$ coincide.*

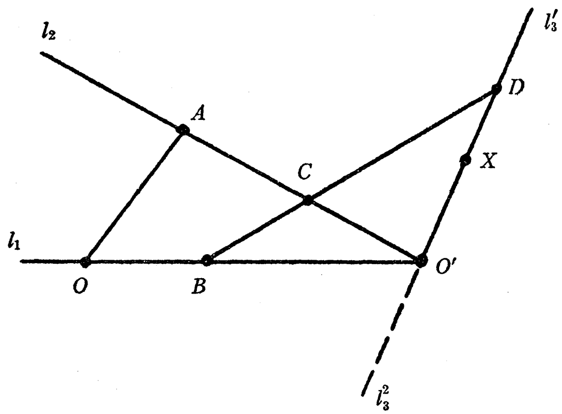

FIG. 6

Proof. Let X be a point of $\{B_{o'}\}$, i.e., a point of a half-line of $B_{o'}$. We want to prove that X also belongs to $\{B_o\}$. As the points of the lines OO', OA, and AO' belong to $\{B_o\}$, it is sufficient to show the existence of two distinct points of the line $O'X$ which belong to $\{B_o\}$. As O' already belongs to $\{B_o\}$, it remains to prove the existence of another point of $O'X$ belonging to $\{B_o\}$.

Let ϕ be the coordinate function for $B_{o'}$ such that $\phi(O'O) = \phi(l_1) = [0]$ and $\phi(O'A) = \phi(l_2) = [\alpha]$, $\alpha < \pi$. Then if we denote by l_3' and l_3^2 the distinct half-lines of $O'X$ with vertex O', $\overline{\phi(l_3')}$ or $\overline{\phi(l_3^2)}$ is between 0 and π (otherwise, X belongs to the line OO' and $X \in \{B_o\}$). Let $\phi(l_3') = [\beta]$, $\beta < \pi$, then either α is between 0 and β, or else β is between 0 and α (otherwise $\alpha = \beta$ so that X is on the line $O'A$ and $X \in \{B_o\}$). Let α be between 0 and β, then

$$([\beta] - [\alpha]) + ([\alpha] - [0]) = ([\beta] - [0]),$$

i.e.,

$$(\phi(l_3') - \phi(l_2)) + (\phi(l_2) - \phi(l_1)) = (\phi(l_3') - \phi(l_1)),$$

so that $[\angle l_3' l_2 + \angle l_1 l_2] = [\angle l_3' l_1]$. The continuity axiom implies that there exist points B, C, D on l_1, l_2, and l_3' respectively which are collinear. As $B, C \in \{B_o\}$, D is the required point, $X \in \{B_o\}$, and $\{B_{o'}\} \subseteq \{B_o\}$. Similarly if β is between 0 and α, $\{B_{o'}\} \subseteq \{B_o\}$. By symmetry $\{B_o\} \subseteq \{B_{o'}\}$.

If two lines of a plane have no point in common, then they are *parallel*.

LEMMA 5. *If B_o is a bundle, P is a point of $\{B_o\}$ different from O, and L is a line defined by a half-line of B_o not containing P, then there exists one and only one line in $\{B_o\}$ containing P and parallel to L.*

The proof of this lemma can be found in [1].

THEOREM 10. *Two planes coincide if and only if they have three noncollinear points in common.*

Proof. Let B_o and $B_{o'}$ be two bundles such that the three noncollinear points A, B, C belong to $\{B_o\} \cap \{B_{o'}\}$. We shall prove that if $X \in \{B_{o'}\}$, then $X \in \{B_o\}$. Suppose that $X \in \{B_{o'}\}$ so that X belongs to a half-line l_1 of $B_{o'}$. At least one of the three points A, B, C, say A, is not on the line l containing l_1. Let L be a line in $\{B_{o'}\}$ containing A and parallel to l. If B and C are not on L, then the lines AB and AC meet l in two distinct points U and V (Fig. 7); as U, $V \in \{B_o\}$, then $X \in \{B_o\}$. If B or C, say B, is on L then C is not on L and the lines AC and BC meet l in two distinct points V and W. V and W being in $\{B_o\}$, $X \in \{B_o\}$. In the same way $\{B_o\} \subseteq \{B_{o'}\}$, and $\{B_o\} = \{B_{o'}\}$.

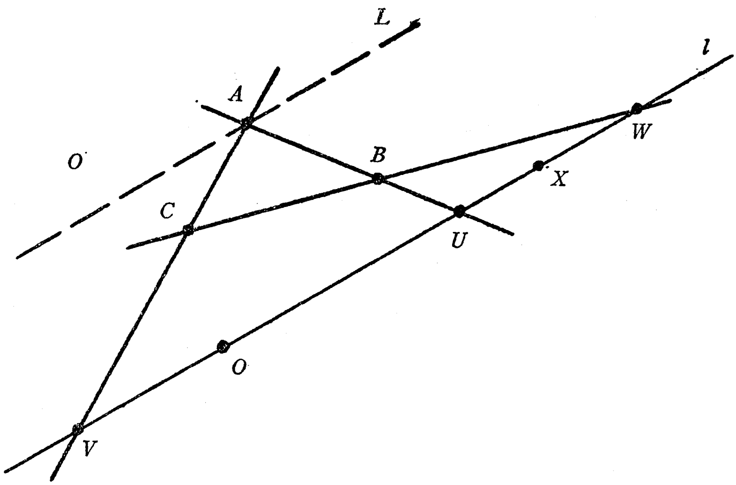

FIG. 7

The results of Lemmas 2 and 5 can now be formulated in a more general form as follows:

THEOREM 11. *In a plane, from a given point not on a given line there exists one and only one perpendicular to that line, and from a given point not on a given line, there exists one and only one parallel to that line.*

The following theorem is an immediate consequence of the continuity axiom.

THEOREM 12. *If three distinct points A, B, and C do not lie on the same line, and if D and E are two points such that C is between B and D, and E is between A and C, then there exists between A and B a point F such that D, E, F lie on the same line.*

8. 3-dimensional euclidean space. The following axiom is now added to the structure.

S. *There exists a point not on a given plane.*

The 3-dimensional euclidean space is introduced as defined by O. Veblen in [4]. A set of four noncoplanar points is called a *tetrahedron* whose *faces* are the interior of the triangles defined by the elements of the tetrahedron (the interior of a proper triangle ABC is defined to be the class of points P between X and Y, where X and Y belong to different intervals defined by the points A, B, and C). A 3-*space* $ABCD$ is the set of all points collinear with any two points of the faces of the tetrahedron $ABCD$.

Except for the Axiom X, the remaining eleven axioms given in [4] are either a property of real numbers, a consequence of a definition given here, or one element of the following list of axioms and theorems: L_1, L_2, L_3, S, Theorem 11, and Theorem 12. The Axiom X says that all points belong to the same 3-space. We have then in [4] the proof of the following property concerning categoricity:

THEOREM 13. *If M_1 and M_2 are two models of a given 3-space, then they are isomorphic.*

I wish to acknowledge my indebtedness to the referee. I owe to him in particular the present formulation of Theorem 2 and its corollaries.

References

1. G. D. Birkhoff, A set of postulate for plane geometry based on scale and protractor, Ann. of Math., 33 (1932) 329–345; reprinted in George David Birkhoff, Collected mathematical papers, vol. III, Amer. Math. Soc., New York (1950), 365–381.

2. K. Borsuk and W. Szmielew, Foundations of Geometry, Amsterdam, 1960.

3. S. MacLane, Metric postulates for plane geometry, this MONTHLY, 66 (1959) 543–555.

4. Oswald Veblen, A system of axioms for geometry, Trans. Amer. Math. Soc., 5 (1904) 343–384.

ON THE INEQUALITY OF KANTOROVICH

E. F. BECKENBACH, University of California, Los Angeles

1. Introduction. For any positive numbers $(x)\equiv(x_1, x_2, \cdots, x_n)$, $n>1$, and positive weights $(\alpha)\equiv(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^{n}\alpha_i=1$, we define the *mean of order* t, $-\infty\leqq t\leqq\infty$, of the numbers (x) with weights (α) by

$$M_t(x;\alpha) = \left(\sum_{i=1}^{n}\alpha_i x_i^t\right)^{1/t}$$

for t finite and $\neq 0$, and otherwise by

$$M_0(x;\,\alpha) \equiv \prod_{i=1}^{n} x_i^{\alpha_i},$$

$$M_{-\infty}(x;\,\alpha) \equiv \min_{i=1}^{n} x_i,$$

$$M_{\infty}(x;\,\alpha) \equiv \max_{i=1}^{n} x_i.$$

The familiar harmonic, geometric, and arithmetic means are included as the special cases $t=-1$, 0, and 1, respectively.

It is easy to show that $\lim_{t\to-\infty} M_t(x;\,\alpha)=M_{-\infty}(x;\,\alpha)$, $\lim_{t\to\infty} M_t(x;\,\alpha) =M_\infty(x;\,\alpha)$, and it readily follows from l'Hospital's rule that

$$\lim_{t\to 0} M_t(x;\,\alpha) = M_0(x;\,\alpha).$$

Thus $M_t(x;\,\alpha)$ is a continuous function of t in the closed interval $[-\infty,\,\infty]$.

If all the x_i are equal, $x_i=x_0$ for $i=1, 2, \cdots, n$, then we have $M_t(x;\alpha) =x_0$ for all t; otherwise, as is well known [**4**, p. 17; **11**, p. 26], $M_t(x;\alpha)$ is a strictly increasing function of t. In particular, this statement implies the classical inequality between the geometric mean $M_0(x;\,\alpha)$ and the arithmetic mean $M_1(x,\,\alpha)$, namely $M_0(x;\,\alpha) \leqq M_1(x,\,\alpha)$, with equality if and only if all the x_i are equal.

In the present paper we are concerned with lower and upper bounds of the ratio $M_s(x;\alpha)/M_r(x;\alpha)$, for arbitrary r and s, $-\infty \leqq r < s \leqq \infty$, and for positive x_i subject to certain additional constraints.

Lower bounds are discussed in Section 2. It follows from the monotonicity property of $M_t(x;\,\alpha)$, mentioned above, that we have

$$\frac{M_s(x;\,\alpha)}{M_r(x;\,\alpha)} \geqq 1 \quad \text{for } r < s,$$

with equality if and only if all the x_i are equal. We do not, however, assume the validity of this result; rather, it appears as a special case (the Corollary to Theorem 1) of a general inequality that is established below, in which some of the x_i are fixed, $x_i=c_i$, $i=1, 2, \cdots, m$, and others are allowed to vary freely, $0<x_i<\infty$, $i=m+1, m+2, \cdots, n$. The proof of Theorem 1 implies a new and, in the author's estimation, illuminating derivation of the much-proved [**5**, p. 54] inequality between the geometric mean and the arithmetic mean.

In Section 3 we turn to a consideration of upper bounds. If the positive variables x_i are not further constrained, so that the ratio of the largest to the smallest x_i is not bounded, then neither is the ratio $M_s(x;\alpha)/M_r(x;\alpha)$ bounded. Accordingly, for given positive numbers A, B, with $0<A<B$, we consider variables x_i constrained by

$$\text{(1)} \qquad 0 < A \leqq x_i \leqq B, \qquad i = 1, 2, \cdots, n.$$

This is a reasonable restriction in many applied problems.

For numbers x_i subject to the above constraint (1), and for $r=-1$, $s=1$, the inequality of Kantorovich [14, 15] gives an upper bound to the mean-value ratio, namely

$$\frac{M_1(x;\alpha)}{M_{-1}(x;\alpha)} \leqq \frac{(A+B)^2}{4AB}.$$

Numerous proofs of the inequality of Kantorovich have been given, and the inequality has been generalized in various ways (see the bibliography at the end of this paper). In particular, an upper bound, including that of Kantorovich as a special case, has been obtained by Cargo and Shisha [6] for arbitrary finite $r<s$.

If, now, some of the x_i are fixed, say $x_i=c_i$, $A\leqq c_i\leqq B$, $i=1, 2, \cdots, m$, and only the rest allowed to vary, then new upper bounds, usually more restrictive than those of Kantorovich and Cargo-Shisha, can be expected. Such upper bounds, including those of Kantorovich and Cargo-Shisha as special cases, are established in Section 3.

The methods of the present paper can be applied to yield analogous extensions of the inequalities of Hölder and Minkowski, as we shall show elsewhere.

2. Lower bound. Letting the symbol $M_t(c, x;\alpha)$ denote the mean of order t and weights (α) of the vector $(c_1, c_2, \cdots, c_m, x_{m+1}, x_{m+2}, \cdots, x_n)$, we now establish the following result.

THEOREM 1. *Let there be given positive weights* $(\alpha)\equiv(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^n \alpha_i=1$, $n>1$, *and positive numbers* $(c)\equiv(c_1, c_2, \cdots, c_m)$, $1\leqq m<n$. *For any positive numbers* $(x)\equiv(x_{m+1}, x_{m+2}, \cdots, x_n)$, *and any indices r and s*, $-\infty\leqq r<s\leqq\infty$, *we have*

$$\frac{M_s(c, x;\alpha)}{M_r(c, x;\alpha)} \geqq \frac{M_s(c, \bar{c};\alpha)}{M_r(c, \bar{c};\alpha)}, \tag{2}$$

where, except for the combination $r=-\infty$, $s=\infty$, *each component* $\bar{c}_j$, $j=m+1, m+2, \cdots, n$, *of* $(\bar{c})$ *is given by*

$$\bar{c}_j=\bar{c}=\begin{cases} \min\limits_{i=1}^{m} c_i, & r=-\infty,\ s \text{ finite}, \\[2ex] \left\{\dfrac{\sum_{i=1}^m \alpha_i c_i^s}{\sum_{i=1}^m \alpha_i c_i^r}\right\}^{1/(s-r)} & r \text{ and } s \text{ finite}, \\[2ex] \max\limits_{i=1}^{m} c_i, & r \text{ finite},\ s=\infty. \end{cases} \tag{3}$$

and where, for $r=-\infty$ and $s=\infty$, each component $\bar{c}_j$ of $(\bar{c})$ is arbitrary, $j=m+1, m+2, \cdots, n$, subject only to

$$\min_{i=1}^{m} c_i \leqq \bar{c}_j \leqq \max_{i=1}^{m} c_i. \tag{4}$$

The values $\bar{c}_j$ given by (3) satisfy (4). Equality holds in (2) if and only if each $x_j=\bar{c}, j=m+1, m+2, \cdots, n$, except that for $r=-\infty$ and $s=\infty$ equality holds if and only if each x_j satisfies

$$\min_{i=1}^{m} c_i \leqq x_j \leqq \max_{i=1}^{m} c_i, \qquad j=m+1, m+2, \cdots, n. \tag{5}$$

Proof. First, to show that the values $\bar{c}_j=\bar{c}$ given by (3) satisfy (4) for r and s finite (the other cases are immediate), let

$$a=\min_{i=1}^{m} c_i, \qquad b=\max_{i=1}^{m} c_i.$$

Then we have

$$\frac{\bar{c}}{a}=\left[\frac{\sum_{i=1}^{m} \alpha_i (c_i/a)^s}{\sum_{i=1}^{m} \alpha_i (c_i/a)^r}\right]^{1/(s-r)}.$$

Since $c_i/a \geqq 1$ and $s>r$, it follows that $(c_i/a)^{s-r} \geqq 1$, $(c_i/a)^s \geqq (c_i/a)^r$, whence we get $\bar{c}/a \geqq 1$, with equality if and only if $c_1=c_2=\cdots=c_m=a=\bar{c}$. Similarly, we obtain $\bar{c}/b \leqq 1$, again with equality if and only if all the c_i are equal, $i=1, 2, \cdots, m$.

For additional properties of the mean-value function $\bar{c}$, see [**3**, **8**].

Define $f(c, x; \alpha; r, s)$, or briefly $f(x)$, by

$$f(x) \equiv \frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)}. \tag{6}$$

For r and s finite and $rs \neq 0$, and for each j, $j=m+1, m+2, \cdots, n$, a computation (cf. [**6**]) gives

$$f_j(x) \equiv \frac{\partial}{\partial x_j} f(x) = P_j Q_j, \tag{7}$$

where

$$P_j \equiv \alpha_j x_j^{r-1}\left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s\right)^{1/(s-1)}\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right)^{-1/(r-1)}, \tag{8}$$

$$Q_j \equiv x_j^{s-r}\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right) - \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s\right). \tag{9}$$

Since $P_j > 0$ for all positive (x), it follows from (7) and (9) that $f_j(x) = 0$ if and only if

$$x_j^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s}{\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r}. \tag{10}$$

Now the right-hand side of (10) is the same for all j, $j = m+1, m+2, \cdots, n$, and accordingly *any* solution $x_{m+1}, x_{m+2}, \cdots, x_n$ of the system of equations

$$f_{m+1}(x) = 0, \quad f_{m+2}(x) = 0, \cdots, f_n(x) = 0, \tag{11}$$

must have all its components equal, say $x_{m+1} = x_{m+2} = \cdots = x_n = \bar{x}$. From (10) we then obtain

$$\bar{x}^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s + \bar{x}^s \sum_{i=m+1}^{n} \alpha_i}{\sum_{i=1}^{m} \alpha_i c_i^r + \bar{x}^r \sum_{i=m+1}^{n} \alpha_i}.$$

Clearly, however, we have

$$\bar{x}^{s-r} = \frac{\bar{x}^s \sum_{i=m+1}^{n} \alpha_i}{\bar{x}^r \sum_{i=m+1}^{n} \alpha_i},$$

whence, from "proportion by division" [24], i.e., from the fact that

$$\frac{a}{b} = \frac{c}{d} \quad \text{implies} \quad \frac{a}{b} = \frac{c-a}{d-b}, \tag{12}$$

provided the denominators do not vanish, we obtain

$$\bar{x}^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s}{\sum_{i=1}^{m} \alpha_i c_i^r}.$$

Hence we have $\bar{x} = \bar{c}$, where $\bar{c}$ is given by (3), whence the system (11) has no solution other than

$$x_{m+1} = x_{m+2} = \cdots = x_n = \bar{c}. \tag{13}$$

That (13) does in fact furnish a solution to the system (11) follows, by (10), from "proportion by composition," wherein the minus signs in (12) are replaced by plus signs.

Geometrically speaking, we have now shown, for r and s finite and $rs\neq 0$, that there is precisely one horizontal tangent hyperplane to the hypersurface

$$S: y = f(x), \qquad 0 < x_i < \infty,$$

in the $(x_{m+1}, x_{m+2}, \cdots, x_n)$-space, and that the point of tangency is the point on S at which each $x_i = \bar{c}$.

We still have to show that there is a minimum value of y on S, namely the corresponding

$$\bar{y} = f(\bar{c}) = \frac{M_s(c, \bar{c}; \alpha)}{M_r(c, \bar{c}; \alpha)}.$$

For any numbers A and B satisfying the inequalities

$$0 < A < \bar{c} < B, \tag{14}$$

consider the $(n-m)$-dimensional cube I_{AB} determined by the inequalities $A \leqq x_i \leqq B$, $i = m+1, m+2, \cdots, n$. Let ℓ be a ray, or half-line, extending perpendicularly from one of the coordinate hyperplanes and intersecting I_{AB}; thus ℓ is determined by

$$x_j > 0, \qquad x_i = x_{i0}, \qquad A \leqq x_{i0} \leqq B, \qquad\qquad i \neq j,$$

with x_j varying, $0 < x_j < \infty$, for some fixed j, $m+1 \leqq j \leqq n$, and with $x_i = x_{i0}$ fixed, $A \leqq x_{i0} \leqq B$, for all $i \neq j$.

Once more applying the principle of proportion by division, from (7)–(10) we see that on ℓ we have

$$f_j(x) = 0$$

at just one point, namely at the point where

$$x_j = x_{j0} = \left\{ \frac{\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{\substack{i=m+1 \\ i\neq j}}^{n} \alpha_i x_{i0}^s}{\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+1 \\ i\neq j}}^{n} \alpha_i x_{i0}^r} \right\}^{1/(s-r)}.$$

We now show that x_{j0} satisfies the inequalities

$$A < x_{j0} < B, \tag{15}$$

as follows. We have

$$\frac{x_{j0}}{A} = \left[\frac{\sum_{i=1}^{m} \alpha_i (c_i/A)^s + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i (x_{i0}/A)^s}{\sum_{i=1}^{m} \alpha_i (c_i/A)^r + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i (x_{i0}/A)^r} \right]^{1/(s-r)}. \tag{16}$$

From (3) we get

$$\sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^s = \left(\frac{\bar{c}}{A}\right)^{s-r} \sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^r;$$

therefore, since $s-r>0$ and $0<A<\bar{c}$, we have

$$\sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^s > \sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^r. \tag{17}$$

Since also $0<A\leqq x_{i0}$, $i\neq j$, it follows that $(x_{i0}/A)^{s-r}\geqq 1$, $i\neq j$, whence

$$\sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i \left(\frac{x_{i0}}{A}\right)^s \geqq \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i \left(\frac{x_{i0}}{A}\right)^r. \tag{18}$$

Hence, by (16), (17), and (18), we have $x_{j0}>A$. Similarly, we obtain $x_{j0}<B$.

The proof in the preceding paragraph could have been somewhat shortened if we had further constrained A and B to satisfy the inequalities

$$0 < A < \min_{i=1}^{m} c_i, \qquad B > \max_{i=1}^{m} c_i,$$

instead of merely the inequalities (14), and this actually would have been sufficient for our present purpose. The weaker hypothesis, however, gives a more precise description of the physical situation.

Now on ℓ we have

$$Q = x_i^{s-r}\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i x_{i0}^r\right) - \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i x_{i0}^s\right), \tag{19}$$

which is a linear function of x_j^{s-r}. Since in (19) the coefficient of x_j^{s-r} is positive, and since $s-r>0$, it follows from (7), (8), and (19) that, on ℓ, f is a strictly decreasing function of x_j for $0<x_j<x_{j0}$, and a strictly increasing function of x_j for $x_{j0}<x_j<\infty$.

By (15), then, on the line segment

$$\ell_{AB} = \ell \cap I_{AB},$$

the function f assumes its maximum value only at one or both end points, and its minimum value only at an interior point, of ℓ_{AB}.

The following observations are immediate consequences of the behavior, as determined above, of the function f on the line ℓ:

(a) The function f assumes its maximum value on I_{AB} at one or more of the vertices (extreme points) of I_{AB}, and at no other points of I_{AB}.

(b) The function f assumes its minimum value on I_{AB} at no boundary point of I_{AB}.

(c) Both the maximum value of f on I_{AB}, and the minimum value of f on the boundary of I_{AB}, increase steadily as $A \to 0$, $0 < A < \bar{c}$, and as $B \to \infty$, $\bar{c} < B < \infty$.

Since there must be a horizontal tangent hyperplane at any interior minimum point of the function f on S, and since there is just one point, $(\bar{c}, \bar{y})$, of S at which there is a horizontal tangent hyperplane, it therefore follows from observation (b), above, that on I_{AB} the function f has a unique minimum, namely at $(x) = (\bar{c})$.

Letting $A \to 0$, $0 < A < \bar{c}$, and $B \to \infty$, $\bar{c} < B < \infty$, we conclude from (6) that, for r and s finite, $rs \neq 0$, and for all positive x_i, $i = m+1, m+2, \cdots, n$, the inequality (2) is valid, with equality if and only if $(x) = (\bar{c})$.

The remaining cases, involving $r = -\infty$, $r = 0$, $s = 0$, and $s = \infty$, can be treated by limiting processes, but then the determination of the conditions under which the strict inequality holds are lost in the analysis unless special devices are used. This situation is exemplified in the extreme case $r = -\infty$, $s = \infty$, in which "$>$" gives way to "$=$" for a continuum of values of the x_i.

Direct methods, however, can still be used. For example, if s is finite, $s > 0$, and $r = 0$, then a computation yields

$$f_j(c, x; \alpha; 0, s) = \frac{\partial}{\partial x_j} f(c, x; \alpha; 0, s) = P_j Q_j,$$

where now

$$(20)\qquad \begin{aligned} P_j &= \alpha_j x_j^{-1} \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s \right)^{1/(s-1)} \left(\prod_{i=1}^{m} c_i^{\alpha_i} \right)^{-1} \left(\prod_{i=m+1}^{n} x_i^{\alpha_i} \right)^{-1}, \\ Q_j &= x_j^s - \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s \right). \end{aligned}$$

As before, though P_j is no longer of the form (8), we still have $P_j > 0$ for all positive (x); and the expression (20) for Q_j can be obtained by substituting $r = 0$ in (9). Hence the preceding analysis applies in this case. The case in which r is finite, $r < 0$, and $s = 0$, is similar.

If r is finite, $r \neq 0$, and $s = \infty$, then

$$f(c, x; \alpha; r, \infty) = \frac{\max (c, x)}{\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r \right)^{1/r}},$$

where

$$\max(c, x) \equiv \max_{i=1,\ j=m+1}^{m,n} (c_i, x_j)$$

Computations now yield

$$f_j = -\alpha_j x_j^{r-1} \max(c, x)\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right)^{-1/(r-1)}$$

for $x_j < \max(c, x)$, and

$$f_j = \left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right)^{-1/(r-1)}\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+j \\ i\neq j}}^{n} \alpha_i x_i^r\right)$$

for $x_j = \max(c, x)$. Hence we have

$$f_j < 0 \qquad \text{for } x_j < \max(c, x),$$
$$f_j > 0 \qquad \text{for } x_j = \max(c, x),$$

so that f is a decreasing function of x_j for $x_j < \max(c, x)$, and an increasing function of x_j for $x_j = \max(x) > \max(c)$. Thus (2) holds in this case, with equality if and only if each

$$x_j = \max_{i=1}^{m} c_i, \qquad j = m+1, m+2, \cdots, n.$$

The case $r=0$, $s=\infty$, and the case $r=-\infty$, s finite, can be treated similarly.

Finally, for

$$f(c, x; \alpha; -\infty, \infty) \equiv \frac{\max(c, x)}{\min(c, x)},$$

we note from (4) that $\min(c, \bar{c}) = \min(c)$, $\max(c, \bar{c}) = \max(c)$. Accordingly, since $\min(c, x) \leqq \min(c)$, $\max(c, x) \geqq \max(c)$, with equality if and only if (5) holds, it follows that (2) is satisfied also for $r = -\infty$, $s = \infty$, with equality if and only if all the x_i satisfy (5).

COROLLARY. *For any positive numbers* $(x) \equiv (x_1, x_2, \cdots, x_n)$, $n > 1$, *and any positive weights* $(\alpha) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^{n} \alpha_i = 1$, *the mean-value function* $M_t(x; \alpha)$ *is a nondecreasing function of* t *for* $-\infty \leqq t \leqq \infty$, *and is strictly increasing unless all the* x_i *are equal.*

Proof. In Theorem 1, let $(c) \equiv (x_1)$ have just one member. Then by (3) and (4) we have, for any r and s, $-\infty \leqq r < s \leqq \infty$, $\bar{c}_j = \bar{c} = x_1$, $j = 2, 3, \cdots, n$, whence $M_r(c, \bar{c}; \alpha) = M_s(c, \bar{c}; \alpha) = x_1$. Substitution in (2) now yields

$$\frac{M_s(x; \alpha)}{M_r(x; \alpha)} \geqq \frac{x_1}{x_1} = 1,$$

or $M_s(x;\alpha) \geqq M_r(x;\alpha)$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

3. Upper bound. Let $\sigma = 1 - \sum_{i=1}^m \alpha_i$, let α_0 be any value satisfying $0 \leqq \alpha_0 \leqq \sigma$, and let the symbol $M_t(c, A, B; \alpha, \alpha_0)$ denote the mean of order t and weights $(\alpha_1, \alpha_2, \cdots, \alpha_m, \sigma - \alpha_0, \alpha_0)$ of the vector $(c_1, c_2, \cdots, c_m, A, B)$.

It should be noted that here either one of the weights $\sigma - \alpha_0$ and α_0 might be 0. We nevertheless retain the definitions of $M_{-\infty}$ and M_∞ given on the first page of this paper, namely

$$M_{-\infty}(c, A, B; \alpha, \alpha_0) = \min(c, A, B), \quad M_\infty(c, A, B; \alpha, \alpha_0) = \max(c, A, B), \tag{21}$$

though now we have only

$$\lim_{t \to -\infty} M_t(c, A, B; \alpha, \alpha_0) \geqq M_{-\infty}(c, A, B; \alpha, \alpha_0),$$

$$\lim_{t \to \infty} M_t(c, A, B; \alpha, \alpha_0) \leqq M_\infty(c, A, B; \alpha, \alpha_0)$$

in place of the former equalities. The definitions (21) have been retained to make the statement of the following Theorem 2 simpler than it otherwise would be; further, with the definition (21) we have

$$\begin{aligned} \lim_{\substack{t \to -\infty \\ \sigma - \alpha_0 \to 0^+}} M_t(c, A, B; \alpha, \alpha_0) &= M_{-\infty}(c, A, B; \alpha, \sigma), \\ \lim_{\substack{t \to \infty \\ \alpha_0 \to 0^+}} M_t(c, A, B; \alpha, \alpha_0) &= M_\infty(c, A, B; \alpha, 0), \end{aligned} \tag{22}$$

and these are the limiting processes with which we are actually concerned.

The following result includes the inequalities of Kantorovich and Cargo-Shisha as special cases.

THEOREM 2. *Let there be given positive numbers A and B satisfying $0 < A < B < \infty$, positive weights $(\alpha) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^n \alpha_i = 1$, $n > 1$, and positive numbers $(c) \equiv (c_1, c_2, \cdots, c_m)$, $0 \leqq m < n$. For any positive numbers $(x) \equiv (x_{m+1}, x_{m+2}, \cdots, x_n)$ satisfying $A \leqq x_j \leqq B$, $j = m+1, m+2, \cdots, n$, and any indices r and s, $-\infty \leqq r < s \leqq \infty$, we have*

$$\frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)} \leqq \frac{M_s(c, A, B; \alpha, \alpha_0)}{M_r(c, A, B; \alpha, \alpha_0)}, \tag{23}$$

where

$$\alpha_0 = \begin{cases} 0 & \text{if } \theta < 0, \\ \theta & \text{if } 0 \leqq \theta \leqq \sigma, \\ \sigma & \text{if } \theta > \sigma, \end{cases} \tag{24}$$

with $\sigma = 1 - \sum_{i=1}^m \alpha_i$, and with θ given by

(25) $$\theta = \frac{1}{s-r}\left[\frac{r\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sigma A^r\right)}{B^r - A^r} - \frac{s\left(\sum_{i=1}^{m} \alpha_i c_i^s + \sigma A^s\right)}{B^s - A^s}\right]$$

for r and s finite, $rs \neq 0$, by respective limiting values of (25) for $r=0$ or $s=0$ and the other finite, by $\theta = 0$ for r finite and $s=\infty$, by $\theta=\sigma$ for $r=-\infty$ and s finite, and by $\theta=\sigma/2$ for $r=-\infty$ and $s=\infty$. For $m=0$, the value θ always satisfies the inequalities

(26) $$0 \leqq \theta \leqq \sigma.$$

Equality holds in (23), for r and s finite, if and only if there is a subset

(27) $$(k_1, k_2, \cdots, k_p), \qquad 0 \leqq p \leqq n-m,$$

of $(m+1, m+2, \cdots, n)$ such that

(28) $$\sum_{i=1}^{p} \alpha_{k_i} = \alpha_0,\ x_{k_i} = B \text{ for } i = 1, 2, \cdots, p, \text{ and } x_j = A$$

for all j in the complement of $(k_1, k_2, \cdots, k_p)$ with respect to $(m+1, m+2, \cdots, n)$; for $r=-\infty$ and s finite if and only if we have (28) and

$$\min_{i=1}^{m} (c_i) = A;$$

for r finite and $s=\infty$ if and only if we have (28) and

$$\max_{i=1}^{m} (c_i) = B;$$

and for $r=-\infty$, $s=\infty$ if and only if we have

$$\min_{i=1, j=m+1}^{m,n} (c_i, x_j) = A, \qquad \max_{i=1, j=m+1}^{m,n} (c_i, x_j) = B.$$

Proof. Let us note first that the observation (a) in Section 2 is a consequence merely of the fact that on 1 the function f first decreases as x_j increases from 0 to x_{j0}, and then increases as x_j increases from x_{j0} to ∞.

From this observation, we see that the function f takes on its maximum on I_{AB} only at certain vertices of I_{AB}. This is true in particular for the Kantorovich and Cargo-Shisha case $m=0$, as we see by considering (x) as the cartesian product of two nonnull factors, say (x_1) and $(x_2, x_3, \cdots, x_n)$; the maximum of f is attained only on the vertices of each of the two factors for any fixed determination of a point in the other factor, and hence only on the vertices of the cartesian product.

We accordingly consider the function $g(u; c, A, B; \alpha; r, s)$, or briefly $g(u)$, defined for r and s finite, $r<s$, $rs\neq 0$, by

$$g(u) \equiv \frac{\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/s}}{\left[\sum_{i=1}^{m} \alpha_i c_i^r + uB^r + (\sigma - u)A^r\right]^{1/r}}, \qquad 0 \leqq u \leqq \sigma, \tag{29}$$

and note that for some u_0, $0 \leqq u_0 \leqq \sigma$, we have

$$\max_{x \in I_{AB}} f(x) = g(u_0) \leqq \max_{0 \leqq u \leqq \sigma} g(u). \tag{30}$$

A computation yields $g'(u) = (u - \theta)N(u)$, where

$$N(u) = \frac{(r - s)(B^r - A^r)(B^s - A^s)\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/(s-1)}}{rs\left[\sum_{i=1}^{m} \alpha_i c_i^r + uB^r + (\sigma - u)A^r\right]^{1/(r+1)}}$$

and θ is given by (25). Clearly we have $N(u) < 0$ for $0 \leqq u \leqq \sigma$, so that

$$g'(u) > 0 \text{ for } u \text{ in } (u < \theta) \cap (0 \leqq u \leqq \sigma),$$
$$g'(u) < 0 \text{ for } u \text{ in } (u > \theta) \cap (0 \leqq u \leqq \sigma).$$

Hence, with α_0 given by (24), we have

$$\max_{0 \leqq u \leqq \sigma} g(u) = g(\alpha_0). \tag{31}$$

Accordingly, for r and s finite, $rs \neq 0$, (23) follows from (6), (30), and (31), with equality if and only if (28) holds.

In the special case $m = 0$, we have $\sigma = 1$, $g(0) = g(\sigma) = 1$, whence (26) follows from Rolle's theorem.

We note incidentally that if there is no subset (27) of $(m+1, m+2, \cdots, n)$ for which $\sum_{i=1}^{p} \alpha_{k_i} = \alpha_0$, then

$$\max_{x \in I_{AB}} \frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)}$$

is attained with $x_{k_i} = B$ either on a set with sum as little less than α_0 as possible or on a set with sum as little more than α_0 as possible; the two values must be computed and compared.

All the foregoing analysis holds also in the case $r = 0$ and s finite, $s > 0$, with limiting values as $r \to 0$ given by

$$g(u) = \frac{\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/s}}{B^u A^{\sigma - u} \prod_{i=1}^{m} c_i^{\alpha_i}}, \qquad 0 \leqq u \leqq \sigma,$$

$$N(u) = \frac{-(B^s - A^s)(\log B - \log A)\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/(s-1)}}{B^u A^{\sigma-u} \prod_{i=1}^{m} c_i^{\alpha_i}}$$

and

$$\theta = \frac{1}{s}\left[\frac{1}{\log B - \log A} - \frac{s\left(\sum_{i=1}^{m} \alpha_i c_i^s + \sigma A^s\right)}{B^s - A^s}\right].$$

Similarly, for r finite, $r<0$, and $s=0$, we have

$$g(u) = \frac{B^u A^{\sigma-u} \prod_{i=1}^{m} c_i^{\alpha_i}}{\left[\sum_{i=1}^{m} \alpha_i c_i^r + uB^r + (\sigma - u)A^r\right]^{1/r}}, \qquad 0 \leqq u \leqq \sigma,$$

$N(u)$ again is negative, and

$$\theta = -\frac{1}{r}\left[\frac{r\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sigma A^r\right)}{B^r - A^r} - \frac{1}{\log B - \log A}\right].$$

The cases in which $r = -\infty$, $s = \infty$, or both, can readily be checked by inspection.

The preparation of this paper was sponsored by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

References

1. F. L. Bauer, A further generalization of the Kantorovich inequality, Numer. Math., 3 (1961) 117–119.

2. F. L. Bauer and A. S. Householder, Some inequalities involving the euclidean condition of a matrix, Numer. Math., 2 (1960) 308–311.

3. E. F. Beckenbach, A class of mean value functions, this Monthly, 57 (1950) 1–6.

4. E. F. Beckenbach and Richard Bellman, Inequalities, J. Springer Verlag, Berlin, 1961.

5. E. F. Beckenbach and Richard Bellman, An introduction to inequalitites, Random House, New York, 1961.

6. G. T. Cargo and O. Shisha, Bounds on ratios of means, J. Res. Nat. Bur. Standards, 66B (1962) 169–170.

7. J. W. S. Cassels, pp. 330 and 340 of G. S. Watson, Serial correlation in regression analysis. I, Biometrika, 42 (1955) 327–341.

8. J. B. Diaz and F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya-G. Szegö, and L. V. Kantorovich, Bull. Amer. Math. Soc., 69 (1963) 416–419.

9. Melvin Dresher, Moment spaces and inequalities, Duke Math. J., 20 (1953) 261–271.

10. W. Greub and W. Rheinboldt, On a generalization of an inequality of L. V. Kantorovich, Proc. Amer. Math. Soc., 10 (1959) 407–415.

11. G. H. Hardy, J. L. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, New York, 1952.

12. Peter Henrici, Two remarks on the Kantorovich inequality, this MONTHLY, 68 (1961) 904–906.

13. A. S. Householder and F. L. Bauer, On certain iterative methods for solving linear systems, Numer. Math., 2 (1960) 55–59.

14. L. V. Kantorovich, Funktsionalnii analyz i prikladnaya matematika, Uspehi Mat. Nauk, (N.S.) 3 (1948) 89–185.

15. ———, Functional analysis and applied mathematics, translated by C. D. Benster and edited by G. E. Forsythe, National Bureau of Standards, Report No. 1509, Washington, D. C., 1952, pp. 106–109.

16. J. Kurschak, Math. Phys. Lapok, 23 (1914) 378.

17. Marvin Marcus and A. H. Cayford, Further results on the Kantorovich inequality, Notices Amer. Math. Soc., 9 (1962) 300.

18. Marvin Marcus and N. A. Khan, Some generalizations of Kantorovich's inequality, Portugal. Math., 20 (1961) 33–38.

19. Morris Newman, Kantorovich's inequality, J. Res. Nat. Bur. Standards, 64B (1959) 33–34.

20. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, J. Springer Verlag, Berlin, 1925, pp. 57, 213–214.

21. A. H. Schopf, On the Kantorovich inequality, Numer. Math., 2 (1960) 344–346.

22. P. Schweitzer, Egy egyenlötlensig az aritmetikai középértékröl (An inequality concerning the arithmetic mean), Mat. Fyz. Lapok, 23 (1914) 257–261.

23. W. G. Strang, On the Kantorovich inequality, Proc. Amer. Math. Soc., 11 (1960) 468.

24. G. A. Wentworth, New school algebra, Ginn and Company, New York, 1898.

A SIMPLIFIED PROOF OF THE DIVERGENCE THEOREM

DJAIRO GUEDES DE FIGUEIREDO, Instituto de Matematica Pura e Aplicada, Rio de Janeiro

1. Introduction. Students of potential theory always wonder how to proceed in order to prove the divergence theorem for some general class of regions in R^n. We think that our proof could supply them with one at the level of advanced calculus. It is Whitney's idea [1] to get the theorem for a general region using partitions of unity and approximations, then reducing it to simpler regions. We use this method here.

First we characterize the class of regions for which the theorem will be proved. A set A is said to have *p-content zero* if for each $\delta>0$ there exist k_δ spheres of radius δ that cover A and such that $k_\delta\delta^p\to 0$ as $\delta\to 0$. A *Gaussian region* is an open connected bounded set V in R^n, whose boundary S is made up of two parts, S_0 and S_1, such that: 1) S_0 is a closed set of zero $(n-1)$-content; 2) for every point x of S_1 there exists a neighborhood $N(x)$ such that $N(x)\cap S_1$ is a regular element of surface; moreover, if we change variables so that x_1 is in the direction of the exterior normal $\nu(x)$ to S_1 at x, then $N(x)\cap S$ is represented by an equation $x_1=h(x_2, \cdots, x_n)$, where h has continuous first order derivatives and if $(x_1, x_2, \cdots, x_n)\in N(x)\cap V$ then $x_1<h(x_2, \cdots, x_n)$.

THEOREM. *Let V be a Gaussian region as defined above. Let $F=(F_1, \cdots, F_n)$ $=F(x)$ be a vector function continuous and bounded in $V\cup S_1$ and with continuous and bounded first order derivatives in V. Then*

$$\int_V \operatorname{div} F dx = \int_{S_1} F\cdot \nu d\sigma, \tag{1}$$

where $\nu=\nu(x)$ is the unit exterior normal to S_1 at x.

2. Proof of the theorem. Let us first establish (1) for some special regions.

LEMMA 1. *Formula* (1) *holds for parallelepipeds V, and F as is Theorem* 1.

This lemma can be proved easily by using iterated integrals.

A point $(x_2, \cdots, x_n)$ will henceforth be denoted by x'.

LEMMA 2. *Let V be the set of points x such that $h(x')<x_1<1$ and $-1<x'<1$, where h has continuous first order derivatives in $-1\leqq x'\leqq 1$. Let A be a curved part of S, i.e., $x_1=h(x')$. The function F is supposed to be continuous in $V\cup S$, with continuous and bounded first order derivatives in V and $F=0$ in some neighborhood of $S-A$. Then* (1) *holds.*

Proof. By the change of variables

$$y_1 = x_1 - h(x'), \qquad y' = x',$$

V goes into $V'=\{y: 0<y_1<1-h(y'),\ -1<y'<1\}$. Let K denote the parallelepiped $0<y_1<1$, $-1<y'<1$. Then the function $G(y)=F(y_1+h(y'), y')$ can be extended to the whole of K by defining $G(y)=0$ in $K-V'$, so that $G(y)$ is continuously differentiable in K. Applying Lemma 1 to G in K we get

$$\int_K \operatorname{div} G dy = -\int_{A'} G_1(y) dy', \tag{2}$$

where A' is the image of A. By examining the effect of the change of variables on the two integrals in (1) we see that the left-hand sides (and right-hand sides) of (1) and (2) coincide. This finishes the proof of Lemma 2.

LEMMA 3. *Suppose that all the conditions of the theorem are fulfilled and, moreover, that $F=0$ in some neighborhood N of S_0. Then* (1) *holds.*

Proof. For every point x in $V-N$ we can find a cube U centered at this point and contained in V. On the other hand, for every point x in $S-N$ we can find a cube U with one side parallel to $\nu(x)$ and such that $U\cap V$ is a region of the type of that in Lemma 2. Since $(V\cup S)-N$ is compact we can find a finite number of such cubes U_j $(j=1, \cdots, p)$ whose union covers this set.

Now we determine p cubes U_j', $(j=1, \cdots, p)$ such that $U_j'\subset U_j$, U_j' has sides parallel to U_j and these cubes also constitute an open covering of $(V\cup S)-N$. A partition of unity for the covering (U_j') is constructed as follows. Let α_j be a C_∞-function such that $\alpha_j=0$ outside U_j', and $\alpha_j>0$ in U_j'. It is clear that the function $\alpha=\sum\alpha_j$ is different from zero on $(V\cup S)-N$. Defining $\beta_j=\alpha_j/\alpha$ we conclude that 1) β_j are C_∞-functions; 2) β_j are equal to zero outside U_j and in a neighborhood of the boundary of U_j; 3) $\sum\beta_j=1$. Using this partition of unity (β_j) we see that (1) in this general case reduces to (1) for the particular cases of Lemma 1 and 2.

Lemma 4. *Let A and B be two open sets in R^n such that dist $(A, B)\geqq d$. Then there exists an infinitely differentiable function $\phi(x)$ such that*

$$\phi(x) = 0 \text{ in } A, \qquad \phi(x) = 1 \text{ in } B$$

and $|\operatorname{grad}\phi(x)|\leqq k/d$, where k is a constant.

Proof. Let $\psi(x)$ be an infinitely differentiable function such that

$$\psi(x) = 0 \quad \text{for} \quad |x| \geqq 1$$

and

$$\int \psi(x)dx = 1.$$

Now we define C as the set of all points x such that $\operatorname{dist}(x, B)<d/2$. It is easily verified that the function ϕ defined by

$$\phi(x) = \left(\frac{d}{2}\right)^{-n}\int \psi\left(\frac{y-x}{d/2}\right)dy,$$

satisfies all the requirements of Lemma 4.

In order to conclude the proof of the theorem we define two sequences of sets

$$A_j = \left\{x: \operatorname{dist}(x, S_0) < \frac{1}{2^j}\right\} \qquad j = 1, 2, \cdots,$$

$$B_j = \left\{x: \operatorname{dist}(x, S_0) > \frac{3}{2^j}\right\} \qquad j = 1, 2, \cdots.$$

Using Lemma 4 we find functions $\phi_j\in C_\infty$ that are 0 in A_j, 1 in B_j and such that

$$|\operatorname{grad}\phi_j(x)| \leqq k2^{j-1}.$$

Now ϕ_jF is a function like F of Lemma 3. By Lemma 3, therefore,

$$\int_V \operatorname{div}(\phi_jF)dx = \int_{S_j}\phi_jF\cdot\nu d\sigma$$

or

$$\int_V \phi_j \operatorname{div} F dx + \int_V \operatorname{grad} \phi_j \cdot F dx = \int_{S_1} \phi_j F \cdot \nu d\sigma.$$

It is easily seen that as j goes to infinity the first integral converges to $\int_V \operatorname{div} F dx$ and the last one to $\int_{S_1} F \cdot \nu d\sigma$, which can be taken as an improper integral.

If we show that

$$\lim_{j\to\infty} \int_V \operatorname{grad} \phi_j dx = 0,$$

the proof will be finished. Using the hypothesis on S_0 we conclude that for each $\delta > 0$ the set A_δ (of the points that are at a distance less than δ from S_0) is covered by k_δ spheres of radius 2δ. So the complement of B_j can be covered by k_j spheres of radius $1/2^{j-2}$, say, and k_j is so related to j that

$$k_j \left(\frac{1}{2^{j-1}}\right)^{n-1} \to 0 \quad \text{as} \quad j \to \infty.$$

Using this we obtain

$$\left| \int_V \operatorname{grad} \phi_j F dx \right| \leqq k2^{j-1} \cdot k_j c \left(\frac{1}{2^{j-2}}\right)^{n-1} = 2^n c k k_j \left(\frac{1}{2^{j-1}}\right)^{n-1},$$

which goes to zero as j tends to infinity. Here c is the volume of the unit sphere.

3. Two remarks about Gaussian regions. First we observe that the property of having p-content zero is invariant under Lipschitz mappings.

Secondly, we may prove that a set A in R^n has zero n-content if and only if it has zero Jordan content.

As a consequence of the separability of R^n, we see that S_1 is made up of a denumerable number of regular surfaces $S_{1,j}$, each one an image by a mapping F_j of an open set A_j in $(n-1)$-dimensional space. In order to take surface integrals in $S_{1,j}$ we have to assume that A_j has Jordan content. This implies that the boundary ∂A_j of A_j has zero Jordan content. If we assume that the mapping F_j can be extended up to the boundary ∂A_j in such a way that F_j is a Lipschitz mapping there, we conclude by the above observations that the boundary of $S_{1,j}$ has zero $(n-1)$-content.

From these remarks we conclude that any region in R^n bounded by a finite number of regular surfaces is a Gaussian region. This contains all regions usually occurring in applications of the divergence theorem.

The author would like to thank Jaak Peetre for mentioning that the simple proof of Lemma 4 is due to Hörmander.

Reference

1. H. Whitney, Geometric Integration Theory, Princeton University Press, 1957.

ON TRANSFORMATIONS IN $\mathbf{R}^n$ AND A THEOREM OF SARD

T. M. FLETT, University of Liverpool

1. An elegant proof of the formula for change of variable in a multiple integral has been given by J. Schwartz [4] (see also Zaanen [5], p. 162) in which the theorem is reduced to the following inequality:

THEOREM A. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable* (1-1) *mapping of D into $\mathbf{R}^n$, and let $J(x)$, the Jacobian determinant of f at x, be nonzero on D. Then for any measurable subset E of D the set $f(E)$ is measurable and*

$$m(f(E)) \leqq \int_E |J(x)|\, dx, \tag{1.1}$$

where m denotes n-dimensional Lebesgue measure.

It is, of course, true that under the hypotheses of Theorem A we have equality in (1.1), but the weaker result stated in Theorem A is sufficient for the proof of the formula for change of variable.

A complement to Theorem A is provided by a theorem of Sard (see, for example, de Rham [3], p. 9) which states

THEOREM B. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, let $J(x)$ be the Jacobian determinant of f at x, and let E_0 be the set of points x of D for which $J(x)=0$. Then $f(E_0)$ is of measure zero.*

Extensions of Theorem B under less restrictive conditions have been given by Rado and Reichelderfer, and in particular it has been shown that Theorem B holds if f is merely differentiable on D ([2], pp. 339, 343). Here, however, we restrict ourselves to the case stated above, which is the case most often used in the theory of differentiable manifolds.

The result of Theorem B is most naturally viewed as an extension of the inequality of Theorem A to the case in which $J(x)$ vanishes at points of D, and indeed Theorem B shows that (1.1) continues to hold in this case. It is also clear that we can remove the hypothesis in Theorem A that f is (1-1), for if f is not (1-1) the integral $\int_E |J(x)|dx$ is equal to the measure of $f(E)$ with multiply-covered volumes being counted multiply. We are therefore led to the following theorem, which contains both Theorems A and B.

THEOREM C. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then for any measurable subset E of D the set $f(E)$ is measurable and*

$$m(f(E)) \leqq \int_E |J(x)|\, dx. \tag{1.2}$$

Theorem C is a simple consequence of theorems of Rado and Reichelderfer [2, p. 363], but these theorems themselves use difficult ideas involved in the algebraic topology of $\mathbf{R}^n$. We can also use Schwartz's elementary proof of Theorem A to deal with the subset of D in which $J(x)\neq 0$ and then appeal to Theorem B to complete the argument. However, since Sard's Theorem B is itself an immediate consequence of Theorem C, it seems worth while to give a direct and elementary proof of Theorem C, and this is the purpose of this note. It will be seen that the proof depends on a simple geometrical inequality in which the form of the result is the same whether $J(x)$ is zero or nonzero, and that this inequality leads easily to a form of (1.2) with the measure m replaced by outer measure (Lemma 5). It is only in the proof of the measurability of $f(E)$ (Lemma 6) that we reduce to the case in which $J(x)\neq 0$, and even here this reduction could be avoided by the use of more difficult ideas.

2. We begin by recalling a few definitions. For any point v_0 of $\mathbf{R}^n$ and any set of n linearly independent vectors $a_1, \cdots, a_n$ in $\mathbf{R}^n$, the *parallelotope P with initial vertex v_0 and edge-vectors $a_1, \cdots, a_n$* is the set of all points of $\mathbf{R}^n$ of the form

$$x = v_0 + \sum_{i=1}^{n} \lambda_i a_i,$$

where $\lambda_1, \cdots, \lambda_n$ are real numbers such that $0 \leqq \lambda_i \leqq 1$, $i=1, \cdots, n$. The point $v_0+\frac{1}{2}\sum_{i=1}^{n} a_i$ is called the *centre* of the parallelotope.

For fixed k the set of those points of P for which λ_k has a fixed value equal to either 0 or 1 is called an *$(n-1)$-dimensional face* (or, briefly, *face*) of P, so that the number of faces of P is $2n$. The point

$$v_0 + \lambda_k a_k + \frac{1}{2} \sum_{\substack{i=1 \\ i\neq k}}^{n} a_i$$

is called the centre of the face.

It is immediate that the parallelotope P with initial vertex at the origin and edge-vectors $a_1, \cdots, a_n$ is the image of the unit cube

$$C = \{x = (x^1, \cdots, x^n) \colon 0 \leqq x^i \leqq 1, \ i = 1, \cdots, n\}$$

under the nonsingular linear transformation $h\colon \mathbf{R}^n \to \mathbf{R}^n$ given by

$$h(x) = h(x^1, \cdots, x^n) = \sum_{i=1}^{n} x^i a_i,$$

and equally the image of the unit cube by any nonsingular linear transformation of $\mathbf{R}^n$ onto itself is a parallelotope of this form. It follows that P is compact, and that the frontier of P is the union of the $2n$ faces of P; also (see, for example, Zaanen [5], p. 160) the n-dimensional measure of P is equal to $|\det(h)| = |\det(a_i^j)|$, where a_i^j is the jth coordinate of a_i; and obviously these last results extend to a parallelotope with any initial vertex.

Throughout the following discussion we use the ordinary Euclidean norm for points of $\mathbf{R}^n$ and the corresponding norm for a linear transformation of $\mathbf{R}^n$ into itself, and we denote the inner product of x and y by $x\cdot y$. We use $A(n)$ to denote a positive constant depending only on n, not necessarily the same on any two occurrences.

3. For our proof of Theorem C we require two simple geometrical inequalities.

LEMMA 1. *Let F be a set in $\mathbf{R}^n$ contained in a hyperplane H, let x_0 be a fixed point of F, and let $\|x-x_0\|\leqq d$ whenever $x\in F$. Let also G be the set of points of $\mathbf{R}^n$ whose distance from F is less than δ. Then G is measurable (since it is open) and*

$$m(G) \leqq 2^n(d+\delta)^{n-1}\delta. \tag{3.1}$$

It is evident that G lies between the two hyperplanes parallel to H and distant δ from it, and to prove (3.1) we construct a parallelotope containing G with two of its faces in these hyperplanes.

By a suitable translation we may suppose that H contains the origin, so that H is an $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$. We can therefore find a unit vector a_1 such that $x\cdot a_1=0$ for all $x\in H$ (i.e. such that a_1 is orthogonal to H), and then we can find vectors $a_2, \cdots, a_n$ such that $\{a_1, a_2, \cdots, a_n\}$ is a complete orthonormal set in $\mathbf{R}^n$. Let now $y\in G$. Since every vector in $\mathbf{R}^n$ can be expressed as a linear combination of the a_i, there exist real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$y - x_0 = \sum_{i=1}^{n} \lambda_i a_i.$$

Further, since the distance of y from F is less than δ, there exists $x\in F$ (possibly identical with y) such that $\|y-x\|<\delta$, and then writing

$$y - x = (y - x_0) - (x - x_0),$$

we obtain $\lambda_1=(y-x_0)\cdot a_1=(y-x)\cdot a_1-(x-x_0)\cdot a_1=(y-x)\cdot a_1$, whence $|\lambda_1|\leqq \|y-x\|\,\|a_1\|=\|y-x\|<\delta$. Also

$$\|y - x_0\| \leqq \|y - x\| + \|x - x_0\| < \delta + d,$$

so that for $i=2, \cdots, n$,

$$|\lambda_i| = |(y - x_0)\cdot a_i| \leqq \|y - x_0\|\,\|a_i\| = \|y - x_0\| \leqq d + \delta.$$

It follows that G is contained in the (fixed) parallelotope with centre x_0 and edge-vectors $2\delta a_1$, $2(d+\delta)a_i$, $i=2, \cdots, n$, and since the measure of this parallelotope is $2^n(d+\delta)^{n-1}\delta|\det(a_i^j)|=2^n(d+\delta)^{n-1}\delta$, the result follows.

LEMMA 2. *Let h be a linear transformation of $\mathbf{R}^n$ into itself, let P be the image by h of the unit cube $C=\{x=(x^1, \cdots, x^n): 0\leqq x^i\leqq 1,\ i=1, \cdots, n\}$, and let Q be the set of points of $\mathbf{R}^n$ whose distance from P is less than δ. Then Q is measurable (since it is open) and $m(Q)\leqq|\det(h)|+A(n)(\|h\|+\delta)^{n-1}\delta$.*

Suppose first that $\det(h)=0$, so that h is singular. In this case P is contained in a hyperplane, and we apply Lemma 1 to $F=P$, taking x_0 to be the image of the centre w_0 of C. Since

$$\|h(w)-h(w_0)\| = \|h(w-w_0)\| \leqq \|h\|\,\|w-w_0\| \leqq \tfrac{1}{2}\sqrt{n}\|h\|$$

whenever $w\in C$, we have $\|x-x_0\|\leqq\frac{1}{2}\sqrt{n}\|h\|$ whenever $x\in P$, whence Lemma 1 gives

$$m(Q) \leqq 2^n(\tfrac{1}{2}\sqrt{n}\|h\|+\delta)^{n-1}\delta \leqq A(n)(\|h\|+\delta)^{n-1}\delta,$$

as required.

Suppose next that $\det(h)\neq 0$. In this case P is a parallelotope with measure $m(P)=|\det(h)|$, and it is therefore enough to prove that the open set $Q\backslash P$ has measure not exceeding $A(n)(\|h\|+\delta)^{n-1}\delta$. Since P is compact, for each $y\in Q\backslash P$ there exists $x\in P$ such that $\|y-x\|$ is equal to the distance of y from P, and evidently x is a frontier point of P, so that x lies on one or more $(n-1)$-dimensional faces of P. Since P has $2n$ such faces and each face is the image by h of a face of C, it is now enough to prove that if B is a face of C and E is the set of points of $\mathbf{R}^n$ whose distance from $h(B)$ is less than δ, then

$$m(E) \leqq A(n)(\|h\|+\delta)^{n-1}\delta. \tag{3.2}$$

To prove this last result we observe that $h(B)$ is contained in a hyperplane, so that we can apply Lemma 1 to $F=h(B)$. We choose x_0 to be the centre of the face $h(B)$ of P, so that $\|x-x_0\|\leqq\frac{1}{2}\sqrt{(n-1)}\|h\|$ whenever $x\in h(B)$, and then Lemma 1 gives

$$m(E) \leqq 2^n(\tfrac{1}{2}\sqrt{(n-1)}\|h\|+\delta)^{n-1}\delta \leqq A(n)(\|h\|+\delta)^{n-1}\delta.$$

This proves (3.2), and completes the proof of Lemma 2.

In the case in which $\det(h)\neq 0$ it is tempting to estimate $m(Q)$ by using the inequality $m(Q)\leqq m(P')$, where P' is the smallest parallelotope containing Q with sides parallel to those of P, but unfortunately the measure $m(P')$ tends to infinity as we approach the singular case, i.e. as $\det(h)$ tends to 0 (this is easily seen from a diagram illustrating the plane case). Most proofs of the change of variable formula in which the estimate of the measure of a parallelotope appears, do in fact use an estimate of the form $m(Q)\leqq m(P')$, and it is for this reason that the hypothesis $\inf|J(x)|>0$ is essential to such proofs.

From Lemma 2 we deduce immediately:

LEMMA 3. *Let C be a closed cube in $\mathbf{R}^n$ with sides parallel to the axes and of length α, let h be a linear transformation of $\mathbf{R}^n$ into itself, and let Q be the set of points of $\mathbf{R}^n$ whose distance from the set $h(C)$ is less than $\alpha\delta$. Then Q is measurable (since it is open) and*

$$m(Q) \leqq m(C)\{|\det(h)| + A(n)(\|h\|+\delta)^{n-1}\delta\}.$$

By applying Lemma 3 to the derivative of a differentiable mapping, we obtain the following result; in this we use the definition of derivative given by Dieudonné ([1], Chapter 8).

LEMMA 4. *Let C be a closed cube in $\mathbf{R}^n$ with centre x_0 and with sides parallel to the axes, let f be a differentiable mapping of C into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then*

$$m^*(f(C)) \leqq m(C)\{ \,|\, J(x_0) \,|\, + A(n)(\|f'(x_0)\| + \eta)^{n-1}\eta \}, \tag{3.3}$$

where $\eta = \sup_{x \in C} \|f'(x) - f'(x_0)\|$ and m^* denotes outer Lebesgue measure.

To prove (3.3) let α be the length of the sides of C, and let P be the image of C by the linear transformation $f'(x_0): \mathbf{R}^n \rightarrow \mathbf{R}^n$. By the mean value theorem applied to the function $f - f'(x_0)$ (cf. [1], (8.6.2)), we have for each x of C

$$\|f(x) - f(x_0) - f'(x_0)(x - x_0)\| \leqq \eta\|x - x_0\| < \eta\alpha\sqrt{n},$$

and this inequality expresses the fact that the point $f(x) - f(x_0) + f'(x_0)(x_0)$ of the translate $f(C) - f(x_0) + f'(x_0)(x_0)$ of $f(C)$ is at a distance less than $\eta\alpha\sqrt{n}$ from the point $f'(x_0)(x)$ of P. It follows that this translate of $f(C)$ is contained in the set of points of $\mathbf{R}^n$ whose distance from P is less than $\eta\alpha\sqrt{n}$, and applying Lemma 3 (and noting that det $(f'(x_0)) = J(x_0)$) we obtain immediately the inequality (3.3).

4. The remainder of the proof of Theorem C is similar to Schwartz's proof of Theorem A (cf. [5], p. 165), but we give it here for the sake of completeness. We divide the proof into two further lemmas. It should be noted that Lemma 5 contains Sard's Theorem B.

LEMMA 5. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then for any measurable subset E of D*

$$m^*(f(E)) \leqq \int_E |\, J(x) \,|\, dx, \tag{4.1}$$

where m^* denotes outer Lebesgue measure.

Suppose first that E is a closed cube C with sides parallel to the axes. Since f' is continuous on C, we can divide C into a finite number of nonoverlapping closed cubes $C_1, \cdots, C_N$ with centers $x_1, \cdots, x_N$ and with sides parallel to the axes such that $\|f'(x) - f'(x_k)\| \leqq \epsilon$ whenever $x \in C_k (k = 1, \cdots, N)$. By Lemma 4, for each cube C_k we have

$$m^*(f(C_k)) \leqq m(C_k)\{ \,|\, J(x_k) \,|\, + A\epsilon \},$$

where A is independent of k, so that also

$$m^*(f(C)) \leqq \sum m^*(f(C_k)) \leqq \sum |\, J(x_k) \,|\, m(C_k) + A\epsilon m(C),$$

the summations being extended over all cubes C_k. When the maximum diameter of the cubes C_k tends to 0 the sum $\sum |J(x_k)| m(C_k)$ tends to the Riemann integral of $|J(x)|$ over C, and since ϵ is arbitrary we therefore obtain

$$m^*(f(C)) \leqq \int_C |J(x)| \, dx, \tag{4.2}$$

and this is (4.1) for $E = C$.

Suppose next that E is a measurable subset of D. Then we can find a set E_1 containing E and with measure equal to that of E such that E_1 is the intersection of a contracting sequence of open sets $O_n \subset D$. If now C is a closed cube contained in D with sides parallel to the axes, then for each fixed n the set $C \cap O_n$ is a countable union of nonoverlapping closed cubes with sides parallel to the axes, and applying (4.2) to each such cube and summing we obtain

$$m^*(f(C \cap O_n)) \leqq \int_{C \cap O_n} |J(x)| \, dx,$$

whence also

$$m^*(f(C \cap E)) \leqq \int_{C \cap O_n} |J(x)| \, dx \tag{4.3}$$

(since $E \subset O_n$). Since J is bounded above on C, the integral on the right of (4.3) is finite, and so tends to $\int_{C \cap E_1} |J(x)| dx$ as n tends to $+\infty$, whence

$$m^*(f(C \cap E)) \leqq \int_{C \cap E_1} |J(x)| \, dx = \int_{C \cap E} |J(x)| \, dx.$$

Since D is a countable union of nonoverlapping cubes such as C, the general result (4.1) follows.

LEMMA 6. *Let D be an open set in $\mathbf{R}^n$, and let f be a continuously differentiable mapping of D into $\mathbf{R}^n$. Then $f(E)$ is measurable for every measurable set $E \subset D$.*

(For a proof under more general hypotheses see Rado and Reichelderfer [2], pp. 337, 214.)

Let $J(x)$ be the Jacobian determinant of f at x. It follows immediately from Lemma 5 that if E_0 is the subset of D where $J(x) = 0$, then $f(E_0)$ has measure 0, so that $m(f(E \cap E_0)) = 0$ for every measurable $E \subset D$. Since $D \backslash E_0$ is open, it is therefore enough to prove the result when $J(x) \neq 0$ on D.

Suppose then that $J(x) \neq 0$ on D, so that f is locally a homeomorphism. The open set D is a countable union of closed cubes, and, by the Heine-Borel theorem, we can cover each of these cubes with a finite number of closed cubes on each of which f is a homeomorphism. Hence D is a countable union of closed cubes C_i on each of which f is a homeomorphism, and since $f(E) = \bigcup f(E \cap C_i)$, it is enough to prove that $f(E \cap C)$ is measurable whenever E is measurable and C is a closed cube in D on which f is a homeomorphism.

If E is closed, so are $E\cap C$ and $f(E\cap C)$, and hence if E is a countable union of closed sets, then $f(E\cap C)$ is measurable. Since any measurable set is the union of a set of measure zero and a set which is a countable union of closed sets, it is now enough to prove that $f(E\cap C)$ is measurable when E is of measure zero, and this follows immediately from Lemma 5. This completes the proof of Lemma 6, and so also of Theorem C.

References

1. J. Dieudonné, Foundations of modern analysis, Academic Press, New York, 1960.
2. T. Rado and P. V. Reichelderfer, Continuous transformations in analysis, Springer-Verlag, Berlin, 1955.
3. G. de Rham, Variétés différentiables, Hermann, Paris, 1955.
4. J. Schwartz, The formula for change in variables in a multiple integral, this Monthly, 61 (1954) 81–85.
5. A. C. Zaanen, An introduction to the theory of integration, North Holland Publ. Co., Amsterdam, 1958.

THE COMPANION MATRIX AND ITS PROPERTIES

LOUIS BRAND, University of Houston

1. Companion matrix. The companion matrix of the polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{1}$$

is defined as

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix} \tag{2}$$

in which the first superdiagonal consists entirely of ones and all other elements above the last row are zeros. The companion matrix of $\lambda+a_1$ is $[-a_1]$. The characteristic equation of A is $\det(A-\lambda I)=0$ or

$$\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1-\lambda \end{vmatrix} = 0.$$

If we multiply columns 2, 3, $\cdots$, n of this determinant by $\lambda, \lambda^2, \cdots, \lambda^{n-1}$ and add them to the first column, all elements of this column become zero except the last which is now $-f(\lambda)$. Since the cofactor of this element is $(-1)^{n+1}$, the characteristic equation of A is

$$\det(A - \lambda I) = (-1)^n f(\lambda) = 0; \tag{3}$$

and since the highest common divisor of all cofactors in $\det(A - \lambda I)$ is clearly 1, (3) is also the minimum equation of A.

THEOREM 1. *The companion matrix of the polynomial* $f(\lambda)$ *has* $f(\lambda)=0$ *for its characteristic and minimum equations.*

The companion matrix is singular when and only when $a_n=0$; for $\det A = (-1)^n a_n$.

The genesis of the companion matrix is evident when one replaces the linear differential equation

$$f(D)x = 0 \qquad (D = d/dt) \tag{4}$$

or the linear difference equation

$$f(E)x_n = 0 \qquad (E = 1 + \Delta) \tag{5}$$

by a system of n linear equations of the first order. In both cases the matrix of the system is the companion of the polynomial $f(\lambda)$.

For example, the differential equation

$$x''' + ax'' + bx' + cx = 0$$

is replaced by the system

$$\begin{aligned} x' &= \qquad\qquad y \\ y' &= \qquad\qquad\qquad\qquad z \\ z' &= -\,cx - by - az \end{aligned}$$

whose matrix is precisely the companion of the polynomial $\lambda^3+a\lambda^2+b\lambda+c$. Similarly the difference equation

$$x_{n+3} + ax_{n+2} + bx_{n+1} + cx_n = 0$$

may be replaced by the system

$$\begin{aligned} x_{n+1} &= \qquad\qquad y_n \\ y_{n+1} &= \qquad\qquad\qquad\qquad z_n \\ z_{n+1} &= -\,cx_n - by_n - az_n \end{aligned}$$

whose matrix is the companion of the same polynomial.

2. Eigenvectors. The equation $f(\lambda)=0$ may be written in the matrix form

$$Ae(\lambda) = \lambda e(\lambda), \tag{6}$$

where the column vector

$$e(\lambda) = (1, \lambda, \lambda^2, \cdots, \lambda^{n-1});$$

for the first $n-1$ equations are the identities $\lambda^i = \lambda^i$ $(i=1, 2, \cdots, n-1)$ and the last is

$$-a_n - a_{n-1}\lambda - \cdots - a_1\lambda^{n-1} = \lambda^n.$$

Thus if λ_i is an eigenvalue of A, equation (6) is valid for $\lambda = \lambda_i$. Moreover, the rank of the matrix $A - \lambda_i I$ is always $n-1$ even when λ_i is a multiple zero of $f(\lambda)$; for the minor of the element $(n1)$ has a determinant of value 1. The eigenvalue λ_i is therefore associated with just *one* eigenvector $e_i = e(\lambda_i)$; two eigenvectors are called *equal* if one is a scalar multiple of the other. We state this result as

THEOREM 2. *If the polynomial $f(\lambda)$ of degree n has m $(\leqq n)$ distinct zeros λ_i then its companion matrix has exactly m independent eigenvectors*

$$e_i = (1, \lambda_i, \lambda_i^2, \cdots, \lambda_i^{n-1}), \qquad i = 1, 2, \cdots, m, \tag{7}$$

associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$.

These eigenvectors are linearly independent since the rank of the $m \times n$ Vandermonde matrix formed from their components is exactly m.

3. Generalized eigenvectors. When the companion matrix has an eigenvalue λ_1 of multiplicity k, λ_1 satisfies the equations

$$f(\lambda) = 0, \; f'(\lambda) = 0, \cdots, f^{(k-1)}(\lambda) = 0.$$

The first of these equations is equivalent to the matrix equation (6); the others are equivalent to matrix equations obtained from (6) by $k-1$ successive differentiations with respect to λ:

$$Ae^{(j)}(\lambda) = \lambda e^{(j)}(\lambda) + je^{(j-1)}(\lambda), \qquad j = 1, 2, \cdots, k-1,$$

where $e^{(0)}(\lambda)$ means $e(\lambda)$. These are equivalent to the system

$$A\frac{e^{(j)}(\lambda)}{j!} = \lambda\frac{e^{(j)}(\lambda)}{j!} + \frac{e^{(j-1)}(\lambda)}{(j-1)!}, \qquad j = 1, 2, \cdots, k-1. \tag{8}$$

Thus when $\lambda = \lambda_1$ is a k-tuple zero of $f(\lambda)$ we have the k equations

$$\begin{aligned} A\, e_1 &= \lambda_1 e_1 \\ A\, e_2 &= \lambda_1 e_2 + e_1, \\ A\, e_3 &= \lambda_1 e_3 + e_2, \\ &\cdots\cdots\cdot \\ A\, e_k &= \lambda_1 e_k + e_{k-1}, \end{aligned} \tag{9}$$

where $e_1 = e(\lambda_1)$ is *the* eigenvector and

$$e_{j+1} = \frac{e^{(j)}(\lambda_1)}{j!}, \qquad j = 1, 2, \cdots, k-1, \tag{10}$$

are, by definition, $k-1$ *generalized eigenvectors* associated with λ_1. Thus with every k-tuple eigenvalue we associate k eigenvectors of which $k-1$ are generalized. We note that the way in which equations (8) are derived guarantees their consistency. Moreover the k vectors

$$\begin{aligned}
e_1 &= (1, \lambda_1, \lambda_1^2, \lambda_1^3, \quad \cdots, \lambda_1^{n-1})\\
e_2 &= (0, 1, 2\lambda_1, 3\lambda_1^2, \quad \cdots, (n-1)\lambda_1^{n-2})\\
e_3 &= (0, 0, 1, \binom{3}{2}\lambda_1, \quad \cdots, \binom{n-1}{2}\lambda_1^{n-3})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
e_k &= (0, 0, 0, 0, \quad \cdots, \binom{n-1}{k}\lambda_1^{n-k})
\end{aligned}$$

are linearly independent since the rank of the $k\times n$ matrix formed from their components is k; for the $k\times k$ determinant on the left has the value 1.

The entire set of n eigenvalues of A, simple and multiple, are now associated with n eigenvectors; m of these, given by (7), are the eigenvectors associated with the m distinct eigenvalues, whereas the $n-m$ remaining are generalized eigenvectors of the type (10). The entire set is linearly independent and may be used to reduce A to its Jordan normal form.

4. Reduction to the Jordan Normal Form. Let the $n\times n$ companion matrix A have the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ written in an order of increasing multiplicity (such as 3, 1, 1, 2, 2, 2) and denote the associated eigenvectors, actual or generalized, by $e_1, e_2, \cdots, e_n$. Since this set is linearly independent they admit a reciprocal set $e^1, e^2, \cdots, e^n$ defined by $e_i\cdot e^j = \delta_i^j$. Form the two $n\times n$ matrices

$$B = (e_1\ e_2 \cdots e_n) \quad \text{from the } n \text{ columns } e_i,$$

$$B^{-1} = \begin{pmatrix} e^1 \\ e^2 \\ \cdot \\ \cdot \\ \cdot \\ e^n \end{pmatrix} \quad \text{from the } n \text{ rows } e^i.$$

Then $B^{-1}AB = J$ will be in the Jordan normal form. The proof is immediate; for

$$B^{-1}AB = B^{-1}(Ae_1 \mid Ae_2 \mid \cdots \mid Ae_n) = (e^iAe_j)$$

has elements e^iAe_j which are precisely the elements of a Jordan matrix by virtue of Equations (9). An example will make this clear.

The companion matrix of $(\lambda+1)(\lambda-1)(\lambda-2)^2=\lambda^4-4\lambda^3+3\lambda^2+4\lambda-4$, namely

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -4 & -3 & 4 \end{bmatrix},$$

has the eigenvalues $\lambda_1=-1$, $\lambda_2=1$, $\lambda_3=2$, $\lambda_4=2$. The associated eigenvectors are

$$\begin{aligned} e_1 &= e(-1) &&= (1, -1, 1, -1) \\ e_2 &= e(1) &&= (1, 1, 1, 1) \\ e_3 &= e(2) &&= (1, 2, 4, 8) \\ e_4 &= e'(2) &&= (0, 1, 4, 12). \end{aligned}$$

The matrix

$$B = (e_1\ e_2\ e_3\ e_4) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 1 \\ 1 & 1 & 4 & 4 \\ -1 & 1 & 8 & 12 \end{bmatrix},$$

and since $Ae_1=-e_1$, $Ae_2=e_2$, $Ae_3=2e_3$, $Ae_4=2e_4+e_3$, the matrix

$$B^{-1}AB = \begin{bmatrix} e^1Ae_i \\ e^2Ae_i \\ e^3Ae_i \\ e^4Ae_i \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

is in the Jordan normal form.

Of course the above procedure will reduce *any* matrix to the Jordan normal form when the generalized eigenvectors are defined by equations (9). In the case of a companion matrix, however, we have at once their explicit form given by (10).

5. Inverse of a Companion Matrix. The polynomial

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d \tag{11}$$

has the companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix}$$

which may be obtained from I by the succession of row operations

$$A = (4 - 3a)(4 - 2b)(4 - 1c)(-4d)(1432)I.$$

Here (1432) denotes a permutation of rows, $(-4d)$ means row 4 times $-d$, and $(4-1c)$ means row 4 minus c times row 1. Hence, if $d \neq 0$, and we take the inverse operations in reverse order,

$$A^{-1} = (1234)(-4/d)(4 + 1c)(4 + 2b)(4 + 3a)I$$

or

$$A^{-1} = \begin{bmatrix} -c/d & -b/d & -a/d & -1/d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus the inverse of any companion matrix can be written down at once. It is related to the companion matrix of the polynomial

$$\lambda^4 + \frac{c}{d}\lambda^3 + \frac{b}{d}\lambda^2 + \frac{a}{d}\lambda + \frac{1}{d} = 0 \tag{12}$$

whose roots are the reciprocals of the roots of (11): The inverse of A is the companion of the polynomial (12) revolved counterclockwise 180° about its center.

This paper was presented at the joint meeting of the Texas Academy of Science and the Mathematical Association at Galveston, December 8, 1961.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-fourth William Lowell Putnam Mathematical Competition held on December 7, 1963, have been determined in accordance with the constitution of the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Michigan State University, East Lansing, Michigan. The members of the team were S. E. Crick, Jr., R. E. Greene and W. A. Webb; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the team were William Kantor, Steven Sperber and Robert Zarrow; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania. The members of the team were Larry Goldstein, Ralph Greenberg and E. Y. Miller; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were A. C. Hindmarsh, Kenneth Kunen and V. S. Poythress; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were J. H. Spencer, Gordon Wassermann and M. H. Weinless; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Lawrence Corwin, Harvard University; S. E. Crick, Jr., Michigan State University; R. E. Greene, Michigan State University; J. H. Spencer, Massachusetts Institute of Technology; and Lawrence Zalcman, Dartmouth College. To each of these a prize of seventy-five dollars is awarded. The William Lowell Putnam Prize Scholarship to Harvard has been awarded to Mr. Crick, who will begin his graduate work in the fall of 1965. The value of this scholarship has been increased to $2500.00 plus tuition ($1520.00), making a total monetary value of $4020.00.

The six persons ranking second highest in the examination, named in alphabetical order, are R. W. Herrick, Oberlin College; Kenneth Kunen, California Institute of Technology; Gilbert Labelle, University of Montreal; Robert Lee, Reed College; E. Y. Miller, University of Pennsylvania; and Josef Sukonick, University of Pennsylvania. To each of these a prize of thirty-five dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Cornell University, Ithaca, New York, the members of the team being A. D. Jette, D. J. Kilbridge and J. T. Litman; Harvard University, Cambridge, Massachusetts, the members of the team being Jeffrey Cheeger, Melvin Hochster and John Mather; University of British Columbia, Vancouver, British Columbia, the members of the team being S. A. Glass, Joanne McWhirter and Bent Petersen; University of Colorado, Boulder, Colorado, the members of the team being J. M. Cushing, D. E. Maurer and R. C. Misare; and the University of Montreal, Montreal, Quebec, the members of the team being Luc Demers, Gaston Giroux and Cecile Mayrand.

Honorable mention is given to the following twenty-five individuals, named in alphabetical order: Bruce Appleby, Massachusetts Institute of Technology; L. G. Brown, Harvard University; N. H. Camien, California Institute of Technology; M. J. Cohen, California Institute of Technology; David Ebin, Harvard University; P. J. Erdelsky, Case Institute of Technology; Daniel Fendel, Harvard University; Gaston Giroux, University of Montreal; W. E. Heierman, Georgia Institute of Technology; R. B. Hodges, Rice University; A. A. Iarrobino, Jr., Massachusetts Institute of Technology; William Kantor, Brooklyn College; Frank Kaplan, City College; William Kennersley, Rensselaer Polytechnic Institute; Gary Luxton, McGill University; Cecile Mayrand, University of Montreal; V. S. Poythress, California Institute of Technology; S. W. Reyner, South Dakota School of Mines; Michael Rolle, Massachusetts Institute of Technology; Michael Schulz, Michigan

State University; R. P. Stanley, California Institute of Technology; J. J. Weinkam, Xavier University, Cincinnati; J. R. Whitney, Michigan State University; Robert Wilson, American University; and Thomas Zaslavsky, City College.

A total of seventeen hundred five contestants from two hundred five colleges and universities entered the competition. Twelve hundred sixty contestants from one hundred ninety-nine colleges and universities (one hundred seventy having teams) participated in the examination on December 7, 1963.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the competition were given the following problems to solve:

Part I

1. (a) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
   (b) Let $P_1, P_2, \cdots, P_{12}$ be the successive vertices of a regular dodecagon. Explain how the three diagonals P_1P_9, P_2P_{11}, and P_4P_{12} intersect.
2. Let $\{f(n)\}$ be a strictly increasing sequence of positive integers such that $f(2)=2$ and $f(mn)=f(m)f(n)$ for every relatively prime pair of positive integers m and n (the greatest common divisor of m and n is equal to 1). Prove that $f(n)=n$ for every positive integer n.
3. Find an integral formula for the solution of the differential equation
$$\delta(\delta-1)(\delta-2)\cdots(\delta-n+1)y=f(x), \qquad x\geqq 1,$$
for y as a function of x satisfying the initial conditions $y(1)=y'(1)=\cdots=y^{(n-1)}(1)=0$, where f is continuous and
$$\delta\equiv x\frac{d}{dx}\cdot$$
4. Let $\{a_n\}$ be a sequence of positive real numbers. Show that
$$\limsup_{n\to\infty} n\left(\frac{1+a_{n+1}}{a_n}-1\right)\geqq 1.$$
Show that the number 1 on the right-hand side of this inequality cannot be replaced by any larger number. (The symbol lim sup is sometimes written $\overline{\lim}$.)
5. (a) Prove that if a function f is continuous on the closed interval $[0,\pi]$ and if
$$\int_0^\pi f(\theta)\cos\theta\,d\theta=\int_0^\pi f(\theta)\sin\theta\,d\theta=0$$
then there exist points α and β such that
$$0<\alpha<\beta<\pi \quad\text{and}\quad f(\alpha)=f(\beta)=0.$$
   (b) Let R be any bounded convex open region in the Euclidean plane (that is, R is a connected open set contained in some circular disk, and the line segment joining any two points of R lies entirely in R). Prove with the help of part (a) that the centroid (center of gravity) of R bisects at least three distinct chords of the boundary of R.
6. Let U and V be any two distinct points on an ellipse, let M be the midpoint of the chord UV, and let AB and CD be any two other chords through M. If the line UV meets the line AC in the point P and the line BD in the point Q, prove that M is the midpoint of the segment PQ.

Part II

1. For what integer a does x^2-x+a divide $x^{13}+x+90$?
2. Let S be the set of all numbers of the form 2^m3^n, where m and n are integers, and let P be the set of all positive real numbers. Is S dense in P?
3. Find every twice-differentiable real-valued function f with domain the set of all real numbers and satisfying the functional equation

$$(f(x))^2-(f(y))^2=f(x+y)f(x-y)$$

for all real numbers x and y.
4. Let C be a closed plane curve that has a continuously turning tangent and bounds a convex region. If T is a triangle inscribed in C with maximum perimeter, show that the normal to C at each vertex of T bisects the angle of T at that vertex. If a triangle T has the property just described, does it necessarily have maximum perimeter? What is the situation if C is a circle? (A convex region is a connected open set such that the line segment joining any two points of the set lies entirely in the set.)
5. Let $\{a_n\}$ be a sequence of real numbers satisfying the inequalities

$$0\leqq a_k\leqq 100a_n \text{ for } n\leqq k\leqq 2n \text{ and } n=1,2,\cdots,$$

and such that the series

$$\sum_{n=0}^{\infty} a_n$$

converges. Prove that

$$\lim_{n\to\infty} na_n=0.$$

6. Let E be a Euclidean space of at most three dimensions. If A is a nonempty subset of E, define $S(A)$ to be the set of all points that lie on closed segments joining pairs of points of A. For a given nonempty set A_0, define $A_n\equiv S(A_{n-1})$ for $n=1,2,\cdots$. Prove that $A_2=A_3=\cdots$. (A one-point set should be considered to be a special case of a closed segment.)

Solutions. Part I

1. (a) Place the squares externally on the sides of the hexagon. Since the angles between adjacent sides of adjacent squares are all equal to 60°, the gaps can be filled with the six equilateral triangles. Since $60°+90°=150°$ the resulting dodecagon is regular.

(b) The three diagonals are concurrent. Let the dodecagon be composed as described in part (a) in such a fashion that P_1P_{12} is the side of a square. The lines P_1P_{12}, P_2P_{12}, $\cdots$, $P_{11}P_{12}$ divide the angle 150° at P_{12} into ten equal angles of 15°. Therefore the angle $P_1P_{12}P_4$ is equal to 45°, P_4P_{12} is a diagonal of the square on P_1P_{12}, and P_1P_9 is the other diagonal. The three lines P_1P_9, P_2P_{11}, P_4P_{12} all pass through the center of this square.

2. Assume that $f(3)=3+p$, where $p\geqq 0$. Then $f(6)=6+2p$, $f(5)\leqq 5+2p$, $f(10)\leqq 10+4p$, $f(9)\leqq 9+4p$, and $f(18)\leqq 18+8p$. Also, $f(5)\geqq 5+p$, $f(15)\geqq 15+8p+p^2$, and $f(18)\geqq 18+8p+p^2$. Consequently, $18+8p+p^2\leqq 18+8p$, and hence $p=0$ and $f(3)=3$. Since $f(6)=6$, $f(n)=n$ for $n\leqq 6$. In general, if $f(n)=n$ for $n\leqq 2k$, where k is an integer >1, $f(2k-1)=2k-1$, and hence $f(4k-2)=4k-2$, and $f(n)=n$ for $n\leqq 4k-2$. Since $4k-2>2k$, induction shows that $f(n)=n$ for all positive integers n.

3. The first step is to show that

$$\delta(\delta-1)(\delta-2)\cdots(\delta-n+1)=x^nD^n.$$

Proof by induction reduces to showing that

$$D^n(\delta-n)=xD^{n+1},$$

which can itself be proved by induction. Alternatively, the initial identity can be proved by showing that it is valid when applied to every nonnegative integral power x^k, and hence valid for every polynomial, and this verification for x^k reduces to showing that

$$\delta(\delta-1)\cdots(\delta-n+1)x^k=k(k-1)\cdots(k-n+1)x^k.$$

The given differential equation becomes $x^nD^ny(x)=f(x)$, and the solution is provided by Liouville's formula for iterated integrals:

$$y(x)=\frac{1}{(n-1)!}\int_1^x(x-t)^{n-1}\frac{f(t)}{t^n}\,dt.$$

4. Assume the conclusion is false. Then there is a positive integer N such that for $n\geqq N$,

$$n\left(\frac{1+a_{n+1}}{a_n}-1\right)<1.$$

This inequality is equivalent to

$$\frac{1}{n+1}<\frac{a_n}{n}-\frac{a_{n+1}}{n+1}.$$

Replacing n by $N, N+1, \cdots, N+k-1$, in turn, and adding the results, we obtain

$$\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{N+k}<\frac{a_N}{N}-\frac{a_{N+k}}{N+k}<\frac{a_N}{N},$$

in contradiction to the divergence of the harmonic series.

To show that 1 cannot be replaced by a larger number, let $a_n=kn$, $n=1, 2, \cdots$. Then

$$n\left(\frac{1+a_{n+1}}{a_n}-1\right)=\frac{1+k}{k}\rightarrow\frac{1+k}{k},$$

which is arbitrarily near 1 for large k. Alternatively, if $a_n=n\log_e n$

$$\lim_{n\to\infty}n\left(\frac{1+a_{n+1}}{a_n}-1\right)=1.$$

5. (a) Since $\sin\theta > 0$ for $0<\theta<\pi$, the second of the two assumed equations implies that $f(\alpha)=0$ for at least one α between 0 and π. Assume now that this α is the *only* zero of f between 0 and π. Then $f(\theta)$ must change sign at α and nowhere else between 0 and π. Hence $f(\theta)\sin(\theta-\alpha)$ is of constant sign and

$$\int_0^\pi f(\theta)\sin(\theta-\alpha)d\theta \neq 0.$$

But this is inconsistent with the assumed vanishing of two integrals.

(b) Choose the center of gravity of the bounded convex domain D as the origin of a system of polar coordinates r, θ. Let $r=r(\theta)$ be the equation of the boundary curve. Obviously, there is at least one direction θ with $r(\theta)=r(\theta+\pi)$. Choose it as the positive x-axis. Put $f(\theta)\equiv r^3(\theta)-r^3(\theta+\pi)$. Since O is the center of gravity, both of the integrals given in part (a) vanish. Hence $f(\theta)$ has at least two zeros, i.e. $r(\theta)=r(\theta+\pi)$ for at least two distinct values of θ with $0<\theta<\pi$.

6. *First solution.* By Steiner's theorem,

$$MPUV \overset{A}{\barwedge} BCUV \overset{D}{\barwedge} QMUV \barwedge MQVU.$$

Hence PQ is a pair of the involution $(MM)(UV)$. Since M is the midpoint of UV, the other invariant point of this involution is the point at infinity, and the involution relates pairs of points equidistant from M.

Second solution. Choose an oblique coordinate system so that the y-axis contains the points U and V, and the x-axis contains the midpoints of chords parallel to U and V. Let the equation of the ellipse be $y^2=ax^2+bx+c$, and those of the lines containing the chords AB and CD be $y=mx$ and $y=nx$, respectively. Then denote:

A: (x_1, y_1), where x_1 is either root of $m^2x^2=ax^2+bx+c$,

C: (x_2, y_2), where x_2 is either root of $n^2x^2=ax^2+bx+c$.

The y-intercept of AC is

$$mx_1 - \frac{y_2-y_1}{x_2-x_1}x_1 = \frac{(m-n)x_1x_2}{x_2-x_1}.$$

With a similar notation, the y-intercept of BD is $(m-n)\bar{x}_1\bar{x}_2/(\bar{x}_2-\bar{x}_1)$. The problem is to show that the sum of these y-intercepts is zero, and this quickly reduces to showing:

$$\frac{x_1+\bar{x}_1}{x_1\bar{x}_1} = \frac{x_2+\bar{x}_2}{x_2\bar{x}_2}.$$

Finally, this follows immediately from the formulas for the sum and product of the roots of a quadratic equation.

Third solution. Replace the ellipse by a circle. Drop perpendiculars a and b from P and Q to AB, and perpendiculars c and d from P and Q to CD. Write $l=UM=MV$, $p=PM$, and $q=MQ$. We wish to prove $p=q$.

We have the following pairs of similar right triangles:

$$\triangle Ma \sim \triangle Mb, \quad \triangle Mc \sim \triangle Md, \quad \triangle Cc \sim \triangle Bb, \quad \triangle Aa \sim \triangle Dd.$$

These yield respectively

$$\frac{p}{q} = \frac{a}{b}, \quad \frac{p}{q} = \frac{c}{d}, \quad \frac{CP}{BQ} = \frac{c}{b}, \quad \frac{PA}{QD} = \frac{a}{d},$$

whence

$$\begin{aligned}\frac{p^2}{q^2} &= \frac{p}{q}\,\frac{p}{q} = \frac{a}{b}\,\frac{c}{d} = \frac{c}{b}\,\frac{a}{d} = \frac{CP}{BQ}\quad\frac{PA}{QD}\\ &= \frac{CP \times PA}{BQ \times QD} = \frac{UP \times PV}{UQ \times QV} = \frac{(l-p)(l+p)}{(l+q)(l-q)} = \frac{l^2-p^2}{l^2-q^2}\\ &= \frac{l^2}{l^2} = 1.\end{aligned}$$

Thus $p=q$, as desired.

Finally, since this is an affine theorem that has been proved for a circle, it holds also for any ellipse.

Solutions Part II

1. $a=2$. The cases $x=0$ and $x=1$ show that a divides 2. The case $x=-1$ shows that a cannot be 1 or -2. The case $x=-2$ shows that a cannot be -1. Finally, $a=2$ can be checked by actual division.

2. Yes. This density is equivalent to the density of the numbers $m \log 2+n \log 3$, which in turn is equivalent to the density of the numbers $m+n(\log 3)/(\log 2)$. Now, $\log 3/\log 2$ is irrational (the proof is easy), and hence the set of all $n \log 3/\log 2$ modulo 1 is dense in the unit interval.

3. Putting $y=x$ shows that $f(0)=0$. Differentiating successively, first with respect to x and then with respect to y, we obtain

$$\begin{aligned}2f(x)f'(x) &= f'(x+y)f(x-y)+f(x+y)f'(x-y),\\ 0 &= f''(x+y)f(x-y)-f(x+y)f''(x-y),\end{aligned}$$

and hence, for all u and v:

$$f''(u)f(v) = f(u)f''(v).$$

There are two main cases: (i) $f''(u)=0$ identically and (ii) there exists a nonempty open interval I in which $f''(u)\neq 0$. Case (i) gives f linear and, since $f(0)=0$, $f(x)=cx$ for some constant c. For case (ii), let v_0 be a point where $f(v_0)f''(v_0)\neq 0$, and let $c=f''(v_0)/f(v_0)$. We now have a nonzero constant c such that $f''(u)=cf(u)$ for all real u. There are two subcases: (iia): $c<0$, (iib): $c>0$. For case (iia), let $c=-a^2$, so that $f''(u)+a^2f(u)=0$, and $f(u)=A \sin au+B \cos au$. Since $f(0)=0$, $B=0$ and $f(u)=A \sin au$. For case (iib), let $c=b^2$, so that $f''(u)-b^2f(u)$

$=0$, and $f(u)=C\sinh bu+D\cosh bu$. As before, $D=0$ and $f(u)=C\sinh bu$. In all cases these solutions check.

4. If the tangent line is permitted to approximate the curve in a neighborhood of a vertex where the normal to the curve does not bisect the angle, the principle of reflection shows easily that the perimeter of the triangle can be increased by a small displacement of the vertex. If an equilateral triangle is "blown up" slightly to give a smooth curve, an inscribed equilateral triangle whose vertices are near the midpoints of the sides of the original triangle has the property described but is certainly not of maximal perimeter. For a circle this property implies that the inscribed triangle is equilateral, and hence of maximal perimeter.

5. By assumption, for any positive integer n, a_{2n} is less than or equal to each of the n numbers $100\,a_n$, $100\,a_{n+1}$, $\cdots$, $100\,a_{2n-1}$, and consequently, as the result of addition and doubling,

$$2na_{2n} \leqq 200(a_n + a_{n+1} + \cdots + a_{2n-1}) \to 0.$$

Similarly, a_{2n-1} is less than or equal to each of the n numbers $100\,a_n$, $100\,a_{n+1}$, $\cdots$, $100\,a_{2n-1}$, and consequently,

$$(2n-1)a_{2n-1} \leqq 2na_{2n-1} \leqq 200(a_n + a_{n+1} + \cdots + a_{2n-1}) \to 0.$$

6. If A_0 is a subset of a line, then A_1 is the smallest interval I containing A_0, and therefore so are A_2, A_3, $\cdots$. If A_0 is a subset of a plane, but not a line, and if u, v, and w are any three points of this plane, define $T(u, v, w)$ to be the smallest convex set containing u, v, and w. If p lies on a segment joining points of the segments $[a, b]$ and $[c, d]$, and if q lies on a segment joining points of the segments $[e, f]$ and $[g, h]$, and if r is a point of the segment $[p, q]$, then r belongs to the smallest convex set containing the points $a, b, \cdots, g, h$, and therefore r belongs to $T(x, y, z)$ for a certain triplet x, y, and z of these points. But $T\subset S(S(\{x, y, z\}))$ and hence $r\in T\subset S(S(A_0))=A_2$. Therefore A_2 is convex, and hence equal to A_3, A_4, $\cdots$. If A_0 is a noncoplanar set, define $T(t, u, v, w)$ to be the smallest convex set containing t, u, v, and w. The procedure is the same as in the plane case, except that $r\in T(s, x, y, z)$ for some four points s, x, y, and z. The inclusion $T\subset S(S(\{s, x, y, z\}))$ follows from the fact that if L_1 and L_2 are two nonadjacent edges of a solid tetrahedron then every point of this solid tetrahedron lies on a segment joining a point of L_1 and a point of L_2.

Mathematical Swifties

"I'm dividing one integer by another," Tom said rationally.
"Why isn't π equal to 22/7?" Tom asked irrationally.
"The ratio of the circumference of a circle to its diameter is not 22/7," said Tom piously.
"The first derivative shows that the function is increasing," Tom stated positively.

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

SOME CONTOUR INTEGRAL SOLUTIONS TO BESSEL'S EQUATION

JAMES M. HORNER, University of Alabama

It is well known that

$$y(z) = z^n \int_C e^{izt}(1 - t^2)^{n-1/2}dt \tag{1}$$

is a solution to Bessel's equation provided that $e^{izt}(1-t^2)^{n+1/2}$ vanishes at the termini of the contour C. This result can be generalized.

THEOREM. *Let $P(t)=at^2+bt+c$, where $b^2-4ac\neq 0$, and let*

$$f(z, t) = A \exp\left[iuzP'(t)\right] + B \exp\left[-iuzP'(t)\right], \tag{2}$$

where A and B are constants and $u=\pm(b^2-4ac)^{-1/2}$. Then

$$y(z) = z^n \int_C f(z, t)P^{n-1/2}dt \tag{3}$$

is a solution of Bessel's equation for appropriate contours C, assuming differentiation of (3) under the sign of integration.

Proof. It is sufficient to show that

$$w_1(z) = \int_C f(z, t)P^{n-1/2}dt \tag{4}$$

satisfies

$$L[w] = zw'' + (2n + 1)w' + zw = 0. \tag{5}$$

For simplicity, let

$$g(z, t) = A \exp\left[iuzP'(t)\right] - B \exp\left[-iuzP'(t)\right]. \tag{6}$$

If (4) is substituted into (5) the result is

$$\begin{aligned} L[w_1] &= z\int_C f(z, t)P^{n-1/2}\{1 + u^2[2P''P - (P')^2]\}dt \\ &\quad + u\int_C P^{n-1/2}\{i(2n + 1)P'g(z, t) - 2zuP''Pf(z, t)\}dt. \end{aligned} \tag{7}$$

But $2P''P-(P')^2=-(b^2-4ac)=-u^{-2}$, so the first integral in (7) vanishes and (7) becomes

$$L[w_1] = 2iu\int_C \frac{\partial}{\partial t}\{g(z, t)P^{n+1/2}\}\,dt. \tag{8}$$

The conclusion then follows for all contours C for which $[g(z, t)P^{n+1/2}]_C=0$.

In particular if r_1 and r_2 are zeros of $P(t)$ then

$$y_n(z) = z^n\int_{r_1}^{r_2} f(z, t)P^{n-1/2}dt \tag{9}$$

is a solution to the Bessel equation when $\text{Re}(n+1/2)>0$. If $A+B=0$, $y_n(z)$ is the trivial solution. When $A+B\neq 0$, $y_n(z)/z^n$ is an integral function for $\text{Re}(n+1/2)>0$, so

$$y_n(z) = K_nJ_n(z). \tag{10}$$

If both members of (10) are divided by z^n and then evaluated at $z=0$, we find that

$$K_n = 2^n(A+B)\Gamma(n+1/2)\int_{r_1}^{r_2} P^{n-1/2}dt. \tag{11}$$

If $P(t)$ is written $P(t)=a(t-r_1)(t-r_2)$ the integral in (11) becomes a Beta function integral, with the change of variable $t=r_1+s(r_2-r_1)$, and

$$K_n = (-a)^{n-1/2}(r_2-r_1)^{2n}(A+B)2^{-n}\sqrt{\pi}\,\Gamma(n+1/2). \tag{12}$$

So we have the following

COROLLARY. *If* $P(t)=a(t-r_1)(t-r_2)$, $r_1\neq r_2$, $a\neq 0$, *then for* $\text{Re}(n+1/2)>0$

$$J_n(z) = \frac{(A+B)^{-1}(2z)^n}{(r_2-r_1)^{2n}\sqrt{\pi}\,\Gamma(n+1/2)}\int_{r_1}^{r_2} f(z, t)[(t-r_1)(r_2-t)]^{n-1/2}dt, \tag{13}$$

where $f(z, t)$ *is given by* (2), *with* $u=\pm a(r_2-r_1)^{-1}$ *and* $A+B\neq 0$.

Reference

1. G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge University Press, 1944.

SOME REMARKS ON ORBITS IN INVERTIBLE SPACES

ANDREW J. UMEN, State University of New York at Buffalo

In a recent note [4] Norman Levine indicated some additional local properties which are necessarily global in invertible spaces. It is the purpose of this note to exhibit some of the properties of orbits in invertible spaces and to relate the study of invertible spaces to an early paper [1] by Richard Arens.

DEFINITION. *If S is an invertible topological space and H the group of all homeomorphisms of S onto S then the symbol $H(x)$ will be used to denote the set of all images of the point x under transformations from H. $H(x)$ is called an orbit in S.*

Hocking and Doyle have pointed out that in an invertible T_1 space each orbit is a nonempty, homogeneous, invertible subspace of S, which is dense in S, and that if $H(x)$ and $H(y)$ are distinct orbits in S then $H(x)\cap H(y)=\emptyset$. Hence, an invertible T_1 space S may be expressed as the union of its orbits:

$$S = \bigcup_{\alpha\in\Lambda} H(x_\alpha)$$

where

$$H(x_\alpha) \cap H(x_\beta) = \emptyset, \quad \text{if } \alpha \neq \beta \text{ and } \overline{H(x_\alpha)} = S$$

for each $\alpha\in\Lambda$.

THEOREM 1. *If S is an invertible T_1 space which is not homogeneous then orbits in S are neither open nor closed.*

Proof. Since S is not homogeneous S has more than one orbit. Suppose some orbit $H(x_\alpha)$ is closed. Then $\overline{H(x_\alpha)}=H(x_\alpha)=S$, contradicting the fact that S has more than one orbit.

On the other hand if some $H(x_\alpha)$ is open, choose $y\notin H(x_\alpha)$. Then $H(y)\cap H(x_\alpha)=\emptyset$ so that $H(y)\subset S-H(x_\alpha)$, a closed set. Hence $S=\overline{H(y)}\subset S-H(x_\alpha)$ which is again a contradiction. It follows immediately from this theorem that,

COROLLARY 1. *In an invertible T_1 space, which is not homogeneous, orbits are not finite.*

COROLLARY 2. *Invertible T_1 spaces, which are not homogeneous, are not finite.*

COROLLARY 3. *In an invertible T_2 space, which is not homogeneous, orbits are not compact.*

THEOREM 2. *In an invertible T_1 space, which is not homogeneous, every orbit is dense in itself.*

Proof. Suppose that $H(x)$ is an orbit and p is a point in $H(x)$. By Corollary 2, p is not the only point in $H(x)$. Suppose that there exists some neighborhood U of p containing no other points of $H(x)$. Since S is invertible there is a homeomorphism $h: S\to S$ such that $h(S-U)\subset U$. This contradicts the assumption that U contains no other points of $H(x)$ beside p.

THEOREM 3. *If S is an invertible T_1 space with a finite number of orbits then any union of orbits is an invertible subspace of S.*

Proof. Suppose that S has $n\geqq 2$ orbits. Any nonempty collection of orbits can be represented as $\hat{S}=S-\bigcup_{i=1}^{k} H(x_i)$ where $k<n$. Suppose that $\hat{U}$ is open in $\hat{S}$. $\hat{U}$ is of the form $U\cap\hat{S}$. Define a mapping $\hat{h}: \hat{S}\to\hat{S}$ as the restriction of the

inverting homeomorphism $h: S \to S$ for U. It is not difficult to show that $\hat{h}$ is a homeomorphism from $\hat{S}$ onto $\hat{S}$ and that $\hat{h}(\hat{S}-\hat{U}) \subset \hat{U}$. Hence $\hat{S}=S-\bigcup_{i=1}^{k} H(x_i)$ is invertible.

THEOREM 4. *If S and T are invertible spaces which are homeomorphic then orbits in T are the homeomorphic images of orbits in S.*

Proof. Let f be a homeomorphism of S onto T and let $H(x)$ be an orbit in S. Let H' be the group of all homeomorphisms of T onto T. The problem is to show that $f(H(x))=H'(f(x))$.

Let $f(y) \in f(H(x))$. Since $y \in H(x)$ there is $h \in H$ such that $h(x)=y$. Consider $fhf^{-1} \in H'$. $fhf^{-1}(f(x))=f(y)$ so that $f(y) \in H'(f(x))$. Hence $f(H(x)) \subset H'(f(x))$.

Now let $y \in H'(f(x))$. Then there is $h' \in H'$ such that $h'(f(x))=y$. Consider $f^{-1}h'f \in H$. We have $f^{-1}h'f(x)=f^{-1}h'h'^{-1}(y)=f^{-1}(y)$. Thus $f^{-1}(y) \in H(x)$ and $y \in f(H(x))$. Hence $H'(f(x)) \subset f(H(x))$.

THEOREM 5. *A subspace S' of an invertible space S is an orbit in S if and only if S' is nonempty, homogeneous, and invariant under transformations from H.*

Proof. Suppose S' is an orbit in S. It follows from previous remarks that S' is nonempty, homogeneous and invariant under H. If on the other hand S' is nonempty, homogeneous and invariant under transformations from H let $x \in S'$. Then $S'=H(x)$. For if $y \in S'$ there is $h \in H$ such that $h(x)=y$. Hence $y \in H(x)$ and $S' \subset H(x)$. If $y \in H(x)$ then there is $h \in H$ such that $h(x)=y$. But $x \in S'$ and since S' is invariant $h(x) \in S'$. Thus $H(x) \subset S'$.

It is not difficult to show that if S and T are invertible spaces then their topological product $S \times T$ is an invertible space. The question naturally arises as to whether the product of orbits will be an orbit in the product space. It is easy to establish that if $H(x)$ is an orbit in S and $H'(y)$ an orbit in T then $H(x) \times H'(y)$ will be a homogeneous subspace of $S \times T$. The following example shows, however, that in general $H(x) \times H'(y)$ may not be invariant in $S \times T$ and hence not an orbit.

Consider the space S consisting of the open interval $(0, 1)$ together with the point 2, where sets will be called open if their complements are countable subsets of $(0, 1)$. This is an invertible space with orbits $(0, 1)$ and $\{2\}$. Now consider the space $S \times S$. The orbits in $S \times S$ are $(0, 1) \times (0, 1)$, $\{(2, 2)\}$ and $(0, 1) \times \{2\} \cup \{2\} \times (0, 1)$. The sets $\{2\} \times (0, 1)$ and $(0, 1) \times \{2\}$ are not orbits in themselves, for it is possible to define a homeomorphism of $S \times S$ onto itself by interchanging the sets $(0, 1) \times \{2\}$ and $\{2\} \times (0, 1)$.

We now consider the problem of topologizing the group H of homeomorphisms of an invertible T_1 space onto itself. Since in an invertible T_1 space we can associate with each nonempty open set U and each proper closed subset C a homeomorphism $h: S \to S$ such that $h(C) \subset U$, it seems appropriate that we should consider subsets of H of the form $W(C, U)$, where $h \in W(C, U)$ provided $h(C) \subset U$.

THEOREM 6. *The collection of all sets of the form* $\bigcap_{i=1}^{n} W(C_i, U_i)$, *where each* C_i *is a proper closed subset and each* U_i *is a nonempty open subset of an invertible* T_1 *space* S, *forms a basis for a topology for* H.

Proof. We note first that no set of the form $W(C, U)$ is empty. Furthermore if $h \in H$ and h is not the identity homeomorphism then h moves some point p and hence $h \in W(p, S-p)$. Now let p be any point in S and U a neighborhood of p; then the set $W(p, U)$ contains the identity homeomorphism. Thus $H = \bigcup_{\alpha \in \Lambda} W(C_\alpha, U_\alpha)$ and the result follows as in [**3**, p. 47–48].

THEOREM 7. *If* S *is a normal invertible* T_1 *space then, with the topology of the preceding theorem,* H *is a topological group.*

Proof. We must show (1) that the mapping $F: H \times H \to H$ defined as $F(f, g) = fg$, the composition of f and g, is continuous and (2) that the mapping $I: H \to H$ defined as $I(h) = h^{-1}$ is continuous. The details are much the same as in [**1**]. To establish (1) let $fg \in W(C, U)$. Then $f(g(C)) \subset U$ and hence $g(C) \subset f^{-1}(U)$. Since S is normal there is an open set V in S such that $g(C) \subset V \subset \overline{V} \subset f^{-1}(U)$. Hence $W(C, V)$ and $W(\overline{V}, U)$ are neighborhoods in H containing g and f respectively. Thus we have $(f, g) \in W(\overline{V}, U) \times W(C, V)$ a neighborhood in $H \times H$. If $(f', g') \in W(\overline{V}, U) \times W(C, V)$ then $F(f', g') = f'g' \in W(C, U)$. Hence F is continuous.

To see that (2) holds we observe that if $h \in W(C, U)$ then $h^{-1} \in W(U', C')$, which is an open set. It follows that I is continuous.

THEOREM 8. *If* S *is a normal invertible* T_1 *space and* $H(x)$ *is an orbit in* S, *then* $H(x)$ *is the* 1-1 *continuous image of the quotient space* H/H_x *where* H_x *is the subgroup of* H *which leaves* x *fixed.*

Proof. Define a mapping $F: H/H_x \to H(x)$ as $F(H_x h) = h(x)$. F is clearly a 1-1 onto mapping. To show that F is continuous let $H_x h$ be an element in the coset space H/H_x and let $U \cap H(x)$ be a neighborhood of $h(x)$ in $H(x)$. Now $W(x, U)$ is a neighborhood of $h \in H$ and hence $H_x W(x, U)$ is a neighborhood of $H_x h$ in the space H/H_x. It follows that $F(H_x W(x, U)) \subset U \cap H(x)$ and thus F is continuous.

The author is indebted to the referee for his helpful suggestions, and wishes also to acknowledge the helpful supervision of Professor E. R. Schneckenburger.

References

1. R. Arens, Topologies for homeomorphism groups, Amer. J. Math., 60 (1946) 593–610.
2. P. H. Doyle and J. G. Hocking, Invertible spaces, this MONTHLY, 68 (1961) 959–965.
3. J. L. Kelley, General topology, Van Nostrand, New York, 1955.
4. N. Levine, Some remarks on invertible spaces, this MONTHLY, 70 (1963) 181–183.

N-TH POWERS IN THE FIBONACCI SERIES

FLOYD BUCHANAN, Buffalo, New York

The two theorems below are concerned with the Fibonacci series or Pisano series, as it is sometimes called. It is the series defined by $U_n = U_{n-1} + U_{n-2}$, U_n being the nth term and $U_1 = U_2 = 1$. The limiting ratio of the terms is equal to the positive root of the quadratic equation $x^2 - x - 1 = 0$ and has an intimate connection with the Golden Ratio of ancient Greek architecture and design. The arrangement of leaf stems on the stalks of plants is another illustration of this series. There are many interesting relations and many striking resemblances to the natural number series.

THEOREM 1. *If p is a prime, U_{p^n} is a prime or product of primes unless $p = 5$, in which case $U_{5^n} = 5^n x$, where x is a prime or product of primes.*

Proof. Let us suppose $U_{p^n} (p \neq 5)$ is divisible by the square of some odd prime q. Then $U_{p^n} = rq^2$, where r is any integer. The only terms in the series which are divisible by any divisor d are those of the form $U_{sj(d)}$, where s is any integer and $j(d)$ is the rank of the first term divisible by d. The notation $j(d)$ is my own, which I adopt for convenience.

By Lucas' theorem on divisibility of terms of this series by odd primes (see [1], p. 396-V), the rank of the first term divisible by q^k, but by no higher power than k of the prime q, is equal to $q^{k-1}j(q) = j(q^k)$. Hence with $k = 2$ we would have $p^n = sj(q^2) = sqj(q)$. It follows that $j(q)$ would be a power of p and that q itself must equal p or simply that $j(p)$ is a power of p.

But if p is a quadratic residue of 5, then $U_{p-1} \equiv 0(p)$ or $p - 1 \equiv 0\ (j(p))$ and if p is a quadratic nonresidue of 5, then $U_{p+1} \equiv 0(p)$ or $p + 1 \equiv 0(j(p))$, (see [1], p. 396-VIII with $\delta = \sqrt{5}$). Also p is a residue of 5 if 5 is a residue of p and vice-versa for nonresidue character. In either case $j(p)$ is prime to p, so $j(p)$ cannot be a power of p and U_{p^n} is not divisible by the square of any odd prime.

If $q = 2$, then $U_{p^n} = 4r$. As $j(4) = 6$, $p^n \equiv 0\ (6)$, which is not possible. Then, for all primes other than 5, the theorem is true.

When $p = 5$, since $j(5) = 5 = U_5$, we have $5^n = 5^{n-1}j(5)$, formally. Evidently $5^n = j(5^n)$ and $U_{5^n} = 5^n x$ where x is prime to 5. If $x = kq^2$ with q any odd prime and k any integer, then $5^n \equiv 0(j(q^2))$ or $5^n \equiv 0(qj(q))$ with the result $q = 5$, but then $x \equiv 0\ (5)$, which is a contradiction.

Our last step is to set $q = 2$ in the preceding paragraph. Then $x = 4k$ and, as $j(4) = 6$, we get $5^n \equiv 0\ (6)$, which is obviously incorrect. Hence, as all possible cases have been rejected, the theorem follows.

THEOREM 2. *U_{12}, U_6 and the trivial $U_1 = U_2 = 1$ are the only terms in the series which are powers of integers other than the first degree.*

Proof. Suppose that $U_n = a^m (m > 1)$, where a is any integer and n is odd with at least two prime factors. With p as any one of the prime factors of n, we can write $U_n = (U_n/U_p)U_p$, and U_n/U_p is an integer since $U_n \equiv 0\ (U_p)$ if $n \equiv 0\ (p)$, (see [1] p. 396-I). Now, if $(U_n/U_p, U_p)=1$, then $U_p=b^m$ or equals 1 if a is a prime. $U_p=b^m$ is not true, by Theorem 1, and $U_p=1$ does not satisfy our conditions. Hence $(U_n/U_p, U_p)=g$, $g>1$. Then $U_p=gu$ and $U_n/U_p=gv$, where $(u, v)=1$. By Theorem 1, $(g, u)=1$, for otherwise U_p has square factors.

Therefore, as $(g^2, u)=1$ and $(u, v)=1$, we have $(g^2v, u)=1$. Now $U_n=(U_n/U_p)U_p=g^2uv=a^m$. Let us put $u=h^m$, $g^2v=i^m$, where $(h, i)=1$. But if $u=h^m$, then $gu=U_p=gh^m$ and U_p would have at least a square factor. So h must equal 1, giving $U_p=g$ and $u=1$.

So, finally, $U_n=a^m=g^2v$ or $U_n=U_p^2v$ with p any odd prime divisor of n. If $p=3$, $n\equiv 0(j(4))$ or $n\equiv 0$ (6) which makes n even, so we will temporarily exclude $p=3$ also. The equation $U_n=U_p^2v$ is a necessary condition but, in this form, it sheds no light on the structure of n. We must transform this equation to a manageable form.

Let $U_p=q_1q_2\cdots q_\tau$, i.e., a product of its prime divisors. Then $U_n\equiv 0(q_i^2)$, $(i=1, 2, \cdots, t)$. For the moment I will exclude $p=5$. Then $n\equiv 0(q_ij(q_i))$. But for all q_i's, $j(q_i)=p$ since $p=sj(q_i)$ is not possible unless $s=1$. We will have $n\equiv 0(pq_i)$ and none of the q_i's equals p since $p=j(p)$ is not possible with p other than 5. Hence $n\equiv 0(pq_1q_2\cdots q_\tau)$. But $q_1q_2\cdots q_\tau=U_p$, so $n\equiv 0(pU_p)$. This equation is necessary and sufficient, for

$$n\equiv 0\ (pq_i)\ \text{ or }\ n\equiv 0\ (q_ij(q_i)=j(q_i^2))$$

so $U_n\equiv 0(q_1q_2\cdots q_\tau)^2$ or $U_n\equiv 0(U_p^2)$.

With the previous exceptions on p we can arrange all the p's that divide n and all of the U_p's in order of magnitude in two lines, one over the other and get a correspondence, term for term, thus:

$$\begin{array}{llll} p_1, & p_2, & p_3 & \cdots \\ U_{p_1}, & U_{p_2}, & U_{p_3} & \cdots. \end{array}$$

Since the U_p's are equal in number to the p's, and are primes or products of primes found only among the p's, and are prime to each other because $(U_x, U_y)=U_{x,y}$, (see [1], p. 396-III), we must have $U_p=p$.

Now $U_p\neq p$ other than $p=5$, which can be very easily seen, and we have excluded $p=5$ so the relation derived from the correspondence cannot hold and our required n's are then only those divisible by 2, 3 or 5 in some combination of powers of these primes. Our last step will be to find the possible ones, consistent with Theorem 1.

We first note, that if U_n is to be divisible by the square of any prime then that prime must occur in n at least in the first power.

If $n\equiv 0$ (25) then U_n is divisible by U_{25}. Now U_{25} is divisible by the prime 3001 and, as 3001 does not occur in n, the prime 5 does not occur in more than the first power. In this case, as $U_{5z}=5x$, where $y\equiv 0$ (2 and 3) and x is prime to 5, the prime 5 obviously would only appear in the first power in U_n. So 5 does not occur at all in n.

If n is divisible by 9, then as $U_9=34$, then n would have to be divisible by 17. So 3 does not occur in more than the first power in n.

If $n\equiv 0$ (8), then as $U_8=21$, $U_n\equiv 0$ (7) so 2 does not occur in n to more than the second power.

Hence, if U_n is to be a power of an integer, we see that n must be a divisor of 12. As $U_6=8$, $U_{12}=144$ and $U_1=U_2=1$, the theorem follows.

Reference

1. L. E. Dickson, History of the Theory of Numbers, Vol. I, Divisibility and Primality, Chelsea, New York.

CONTINUOUS DEPENDENCE OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

AARON STRAUSS, University of Wisconsin

Consider the system of differential equations

$$x' = f(t, x), \tag{E}$$

where x and f are real column n-vectors and t is a real scalar. We follow the notation of [1]. Thus $|x| = \sum_{i=1}^{n} |x_i|$, where $x=(x_1, \cdots, x_n)$, while $f\in(C, \text{Lip})$ in D means that f is continuous in the pair (t, x) in D and that there exists a constant $k>0$ such that for every (t, x_1) and (t, x_2) in D,

$$|f(t, x_1) - f(t, x_2)| \leqq k|x_1 - x_2|. \tag{1}$$

It is known (see [1], Chapter 1) that if $f\in(C, \text{Lip})$, then any solution of (E) is a continuous function of its initial conditions. The standard proof uses successive approximations. Our purpose is to give a new proof, which is more direct and seems more natural. We note that similar methods have been used before to prove weaker theorems (cf. [2]).

THEOREM 1. *Let $f\in(C, \text{Lip})$ in a domain D of the $(n+1)$-dimensional (t, x) space, and suppose that ψ is a solution of* (E) *on some interval $a\leqq t\leqq b$. Define $U_\delta=\{(t, x)\in D: a<t<b, |x-\psi(t)|<\delta\}$. Then for any $\epsilon>0$, there exists a $\delta>0$ such that for any $(\tau, \xi)\in U_\delta$, there is a (unique) solution ϕ of* (E), *such that*

(i) $\phi(\tau)=\xi$,

(ii) *ϕ is defined on all of $a\leqq t\leqq b$,*

(iii) $|\psi(t)-\phi(t)|<\epsilon$ *on $a\leqq t\leqq b$.*

Proof. Let $\delta_1>0$ be such that $U_{\delta_1}\subseteq D$, let $0<\epsilon\leqq\delta_1$, and choose $\delta<\epsilon e^{-k(b-a)}$. Let (τ, ξ) be any point in U_δ and let ϕ be that (local) solution of (E) for which $\phi(\tau)=\xi$. Let $\psi(\tau)=\bar{\xi}$. Then for $\tau\leqq t\leqq b$,

$$\psi(t) = \bar{\xi} + \int_\tau^t f(s, \psi(s))ds.$$

Also for as long as $(t, \phi(t))$ remains in D,

$$\phi(t) = \xi + \int_\tau^t f(s, \phi(s))ds.$$

Therefore

$$\begin{aligned}|\psi(t) - \phi(t)| &\leqq |\bar{\xi} - \xi| + \int_\tau^t |f(s, \psi(s)) - f(s, \phi(s))| \, ds \\ &\leqq \delta + \int_\tau^t k|\psi(s) - \phi(s)| \, ds.\end{aligned}$$

Using Gronwall's inequality (see [1], Chapter 1, problem 1)

$$|\psi(t) - \phi(t)| \leqq \delta e^{k(b-a)} < \epsilon.$$

Thus $(t, \phi(t))$ cannot leave D, and ϕ can be continued to $\tau \leqq t \leqq b$, where $|\psi(t) - \phi(t)| < \epsilon$. A similar argument gives the same result for $a \leqq t \leqq \tau$, proving Theorem 1.

The following corollary of Theorem 1 gives the desired continuity result.

COROLLARY. *Let f and D be as in Theorem* 1. *Let* $(\tau_0, \xi_0) \in D$ *and let* $\psi = \psi(t, \tau_0, \xi_0)$ *be that solution of* (E) *on some interval* $a \leqq t \leqq b$ *for which* $\psi(\tau_0, \tau_0, \xi_0) = \xi_0$. *Then for any* $t_0 \in [a, b]$, ψ *is continuous at* (t_0, τ_0, ξ_0).

Proof. Fix any $t_0 \in [a, b]$. Let δ_1, ϵ, and δ be as in the proof of Theorem 1. Then there exists a $\delta_2 > 0$ such that $|\psi(t', \tau_0, \xi_0) - \psi(t'', \tau_0, \xi_0)| < \delta/4$ whenever $|t' - t''| < \delta_2$, uniformly for $t', t'' \in [a, b]$. Choose $\eta = \min(\delta_2, \delta/4)$. Let (t_1, τ_1, ξ_1) be any point such that $t_1 \in [a, b]$, $(\tau_1, \xi_1) \in D$, and

$$|t_0 - t_1| + |\tau_0 - \tau_1| + |\xi_0 - \xi_1| < \eta.$$

Let $\phi = \phi(t, \tau_1, \xi_1)$ be that solution of (E) for which $\phi(\tau_1, \tau_1, \xi_1) = \xi_1$. We shall show $|\psi(t_0, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| < \epsilon$. Now

$$\begin{aligned}|\xi_1 - \psi(\tau_1, \tau_0, \xi_0)| &\leqq |\xi_1 - \xi_0| + |\xi_0 - \psi(\tau_1, \tau_0, \xi_0)| \\ &\leqq |\xi_1 - \xi_0| + |\psi(\tau_0, \tau_0, \xi_0) - \psi(\tau_1, \tau_0, \xi_0)| \\ &< \delta/4 + \delta/4 = \delta/2.\end{aligned}$$

Thus $(\tau_1, \xi_1) \in U_{\delta/2}$ so that by Theorem 1, we actually have

$$|\psi(t, \tau_0, \xi_0) - \phi(t, \tau_1, \xi_1)| < \epsilon/2 \qquad \text{for } a \leqq t \leqq b.$$

Finally

$$\begin{aligned}&|\psi(t_0, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| \\ &\quad \leqq |\psi(t_0, \tau_0, \xi_0) - \psi(t_1, \tau_0, \xi_0)| + |\psi(t_1, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| \\ &\quad \leqq \epsilon/2 + \delta/4 < \epsilon,\end{aligned}$$

completing the proof.

If f contains a k-dimensional parameter μ, define

$$(\mathrm{E}_\mu)\qquad \begin{aligned} x' &= f(t, x, \mu), \\ I_\mu &= \{\mu: |\mu - \mu_0| < c\} \qquad \text{for some } \mu_0,\ c > 0, \\ D_\mu &= \{(t, x, \mu): (t, x) \in D,\ \mu \in I_\mu\}. \end{aligned}$$

THEOREM 2. *Let $f \in (C, \text{Lip})$ uniformly in μ in D_μ, and suppose that ψ is a solution of (E_{μ_0}) on $a \leqq t \leqq b$. Define*

$$V_\delta = \{(t, x, \mu) \in D_\mu: a < t < b,\ |x - \psi(t)| + |\mu - \mu_0| < \delta\}.$$

Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any point $(\tau, \xi, \mu) \in V_\delta$, there is a (unique) solution ϕ of (E_μ) satisfying (i), (ii), *and* (iii) *of Theorem 1.*

Proof. The procedure is the same as in Theorem 1. Given δ_1 and ϵ, pick $\delta_2 > 0$ such that $|f(t, \psi(t), \mu) - f(t, \psi(t), \mu_0)| < \epsilon[2(b-a)e^{k(b-a)}]^{-1}$ uniformly for $t \in [a, b]$ whenever $|\mu - \mu_0| < \delta_2$. We then choose

$$\delta < \min\left(\delta_2, \frac{\epsilon}{2} e^{-k(b-a)}\right),$$

let $(\tau, \xi, \mu) \in V_\delta$, and let ϕ satisfy (E_μ) through (τ, ξ). Then for $(t, \phi(t))$ in D,

$$\begin{aligned} |\psi(t) - \phi(t)| &\leqq |\bar{\xi} - \xi| + \int_\tau^t |f(s, \psi(s), \mu_0) - f(s, \phi(s), \mu)|\, ds \\ &\leqq |\bar{\xi} - \xi| + \int_\tau^t |f(s, \psi(s), \mu_0) - f(s, \psi(s), \mu)|\, ds \\ &\qquad + \int_\tau^t |f(s, \psi(s), \mu) - f(s, \phi(s), \mu)|\, ds \\ &\leqq \delta + \frac{\epsilon}{2} e^{-k(b-a)} + \int_\tau^t k|\psi(s) - \phi(s)|\, ds. \end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned} |\psi(t) - \phi(t)| &\leqq \left[\delta + \frac{\epsilon}{2} e^{-k(b-a)}\right] e^{k(b-a)} \\ &< \epsilon. \end{aligned}$$

As before, ϕ can be continued to $a \leqq t \leqq b$, where the above inequality remains valid, proving Theorem 2.

It is again an easy consequence that a solution ψ of (E_{μ_0}) through (τ_0, ξ_0) is continuous in the $(n+k+2)$-tuple $(t, \tau_0, \xi_0, \mu_0)$.

Although this method gives the result for the case where, in (1), k is generalized to a continuous (or L_1) function $k(t)$, it is highly unlikely that it can be used to prove the more general "continuous dependence" theorems, as given in Chapter 2 of [1].

The author holds a NSF Coop fellowship. The preparation of this manuscript was partly supported by the NSF (G 24335).

References

1. E. A. Coddington and Norman Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.

2. Witold Hurewicz, Lectures on ordinary differential equations, Wiley and MIT Press, New York, 1958.

ANOTHER PROOF OF WEDDERBURN'S THEOREM

T. J. KACZYNSKI, Evergreen Park, Illinois

In 1905 Wedderburn proved that every finite skew field is commutative. At least seven proofs of this theorem (not counting the present one) are known. See [1], [2], [5] (Part Two, p. 206 and Exercise 4 on p. 219), [6] (two proofs), and [7]. Unlike these proofs, the proof to be given here is group-theoretic, in the sense that the only non-group-theoretic concepts employed are of an elementary nature.

LEMMA. *Let q be a prime. Then the congruence $t^2+r^2\equiv -1 \pmod q$ has a solution t, r with $t\not\equiv 0 \pmod q$.*

Proof. If -1 is a quadratic residue, take $r=0$ and choose t appropriately. Assume -1 is a nonresidue. Then any nonresidue can be written in the form $-s^2 \pmod q$ with $s\not\equiv 0$. If t^2+r^2 is ever a nonresidue for some t, r, set $t^2+r^2 \equiv -s^2$, and we have $(ts^{-1})^2+(rs^{-1})^2\equiv -1$. (Throughout this note, x^{-1} denotes that integer for which $xx^{-1}\equiv 1 \pmod q$.) On the other hand, if t^2+r^2 is always a residue, then the sum of any two residues is a residue, so $-1\equiv q-1=1+1+\cdots+1$ is a residue, contradicting our assumption.

Proof of the theorem. Let F be our finite skew field, F^* its multiplicative group. Let S be any Sylow subgroup of F^*, of order, say, p^α. Choose an element g of order p in the center of S. If some $h\in S$ generates a subgroup of order p different from that generated by g, then g and h generate a commutative field containing more than p roots of the equation $x^p=1$, an impossibility. Thus S contains only one subgroup of order p and hence is either a cyclic or a generalized quaternion group ([3] p. 189).

If S is a generalized quaternion group, then S contains a quaternion subgroup generated by two elements a and b, both of order 4, where $ba=a^{-1}b$. Now a^2 generates a commutative field in which the only roots of the equation $x^2=1$ or $(x+1)(x-1)=0$ are ± 1, so since $(a^2)^2=1$, we have

$$a^2 = -1. \tag{1}$$

Hence $a^{-1}=a^3=-a$, so

$$ba = -ab. \tag{2}$$

Similarly,

$$b^2 = -1. \tag{3}$$

Taking q = characteristic of F $(q\cdot 1=0)$, choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(t+ra+b)(r^2+1+rta+tb) = r(t^2+r^2+1)a + (t^2+r^2+1)b = 0.$$

One of the factors on the left must be 0, so for some numbers $u, v, w, u \not\equiv 0 \pmod q$, we have $w+va+ub=0$, or $b=-u^{-1}va-u^{-1}w$. So b commutes with a, a contradiction. We conclude that S is not a generalized quaternion group, so S is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable ([4], pp. 181–182). Let Z be the center of F^* and assume $Z \neq F^*$. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z (with $Z \subset A$) be a minimal normal subgroup of F^*/Z. A/Z is an elementary abelian group of order p^k (p prime), so since the Sylow subgroups of F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F^*, y any element of A. Since A is normal, $xyx^{-1} \in A$, and $(1+x)y=z(1+x)$ for some $z \in A$. An easy manipulation shows that $y-z=zx-xy=(z-xyx^{-1})x$.

If $y-z=z-xyx^{-1}=0$, then $y=z=xyx^{-1}$, so x and y commute. Otherwise, $x=(z-xyx^{-1})^{-1}(y-z)$. But A is abelian, and z, y, $xyx^{-1} \in A$, so x commutes with y. Thus we have proven that A is contained in the center of F^*, a contradiction.

References

1. E. Artin, Über einen Satz von Herrn J. H. M. Wedderburn, Abh. Math. Sem. Hamburg, 5 (1927) 245.
2. L. E. Dickson, On finite algebras, Göttingen Nachr., 1905, p. 379.
3. M. Hall, The theory of groups, Macmillan, New York, 1961.
4. Miller, Blichfeldt and Dickson, Theory and applications of finite groups, Wiley, New York, 1916.
5. B. L. van der Waerden, Moderne Algebra, Ungar, New York, 1943.
6. J. H. M. Wedderburn, A theorem on finite algebras, Trans. Amer. Math. Soc., 6 (1905) 349.
7. E. Witt, Über die Kommutativität endlicher Schiefkörper, Abh. Math. Sem. Hamburg, 8 (1931) 413.

A NOTE ON PRODUCT SYSTEMS OF SETS OF NATURAL NUMBERS

T. G. McLaughlin, University of California at Los Angeles

In this note, we apply a slight twist to a trick exploited about twelve years ago by J. C. E. Dekker ([2]), our purpose being to expose a couple of elementary facts about nonempty, countable "product systems" of infinite sets of natural numbers which are, at the same time, "finite symmetric difference systems." We proceed in terms of the following definitions.

Definition. *By a product system of subsets of N (N the natural numbers), we mean a collection of subsets of N which contains, along with any two of its members, their intersection.*

DEFINITION. *By a finite symmetric difference system of subsets of* N, *we mean a collection* C *of subsets of* N *such that if* Δ, $\Sigma \subseteqq N$, $\Delta \in C$, *and* $(\Delta-\Sigma)\cup(\Sigma-\Delta)$ *is finite, then* $\Sigma \in C$.

DEFINITION. *Let* $\Delta \subseteqq N$, *and let* C *be a collection of subsets of* N. *We say that* Δ *adheres to* C *(abbr.:* Δ *ad* C*) just in case, for all* $\Sigma \in C$, $\Delta-\Sigma$ *is a finite set.*

Remark. The reader will find in [2] and [3] examples and applications of the notion of a nonempty, countable product system C of infinite subsets of N, with adherent subsets of N.

THEOREM. *Let* C *be a nonempty countable collection of infinite subsets of* N *which is both a product system and a finite symmetric difference system. Then there exists a subset* Δ *of* N *which (modulo finite extensions) is maximal with respect to adherence to* C, *if and only if* C *itself contains a set* Δ *which adheres to* C *(i.e., if and only if* C *is "self-adherent").*

Proof. (1) The trivial direction. Let Δ be an element of C which adheres to C. Then obviously any subset of N which adheres to C can extend Δ by only finitely many numbers.

(2) The less trivial direction. Assume that a certain subset Δ of N adheres to C, and that any subset Σ of N which also adheres to C has only finitely many members not in Δ. Suppose $\Delta \notin C$. Then, since Δ adheres to C and C is a finite symmetric difference system, the following must be true: for each $\Sigma \in C$, $\Sigma-\Delta$ is infinite. Let C be enumerated (possibly with repetitions) as $C=\{A_0, A_1, \cdots\}$; then, since C is a product system, each of the sets $A_0'=A_0-\Delta$, $A_1'=(A_0\cap A_1)-\Delta$, $A_2'=(A_0\cap A_1\cap A_2)-\Delta$, $\cdots$ must be infinite. For each natural number i, let $a_{i0}, a_{i1}, a_{i2}, \cdots$ be an enumeration of A_i'. Define a sequence $\{d_i\}$ as follows:

$$d_0 = a_{00};$$

$$d_1 = a_{1j_1}, \text{ where } j_1 \text{ is the least number } j \text{ such that } a_{1j} > a_{00};$$

$$d_2 = a_{2j_2}, \text{ where } j_2 \text{ is the least number } j \text{ such that } a_{2j} > a_{1j_1};$$

$$\cdots.$$

Then, $\Delta^*=\{d_i \mid i=0, 1, 2, \cdots\}$ is an infinite subset of $N-\Delta$ which (as is very easily verified) adheres to C. Thus $\Delta\cup\Delta^*$ is an infinite extension of Δ which adheres to C; and so we have a contradiction from which the theorem follows.

Remark. It will be noticed that we did not use the whole of our assumption that C is a finite symmetric difference system; the theorem remains true (and the foregoing proof valid) if we replace the finite symmetric difference hypothesis by one which asserts merely that if $\Sigma \in C$ and $\Sigma-\Delta_0$ is finite, where Δ_0 is (modulo finite extensions) maximal adherent to C, then $\Delta_0 \in C$.

We exhibit next a corollary to the above theorem, involving the notion of *recursive function*. For a formal treatment of the concept of recursive function, the reader is referred to [1]. By a *recursive permutation* is meant a recursive function, from N *onto* N, which is one-to-one. A *recursive set* of numbers is a subset Δ of N such that each of Δ, $N-\Delta$ is empty or is the range of a recursive function. It is an elementary result that if Δ, Σ are two infinite, coinfinite recursive sets, there exists a recursive permutation mapping Δ onto Σ.

COROLLARY. *Let C be as in the theorem, with the additional property that C is closed under recursive permutations. Suppose that there exists an infinite subset Δ of N which is not immune (i.e., Δ has an infinite recursive subset), which adheres to C, and which is (modulo finite extensions) maximal with respect to adherence to C. Then C consists entirely of cofinite sets of numbers.*

Proof. Suppose that C contains a noncofinite set. Then, since Δ adheres to C, Δ is noncofinite. By the theorem, Δ itself belongs to C. By hypothesis, Δ has an infinite recursive subset Σ. By a result cited in the paragraph preceding the statement of the corollary, there is a recursive permutation, g, such that $g(\Sigma)=N-\Sigma$, $g(N-\Sigma)=\Sigma$. Since C is closed under recursive permutations, $g(\Delta)\in C$. Hence, $\Delta-g(\Delta)$ is finite. But, obviously, $\Delta-g(\Delta)$ is infinite; and from this contradiction the corollary follows.

Remark. Our proof of the corollary has points in common with the last half of the proof of Theorem 6.5 in [4].

This note was written while the author held a Cooperative Fellowship from the National Science Foundation.

References

1. Martin Davis, Computability and unsolvability, McGraw-Hill, New York, 1958.
2. J. C. E. Dekker, The constructivity of maximal dual ideals in certain Boolean algebras, Pacific J. Math., 3 (1953) 73–101.
3. J. C. E. Dekker and J. R. Myhill, Recursive equivalence types, University of California Press, 1960.
4. J. C. E. Dekker and J. R. Myhill, Some theorems on classes of recursively enumerable sets, Trans. Amer. Math. Soc., 89 (1958) 25–59.

CORRECTION

In the note "On Simultaneous Hermitian Congruence Transformations of Matrices," by K. N. Majindar, published in this MONTHLY, 70 (1963) page 844 the matrix A should be

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \text{ instead of } \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md.

STABILITY BY FRESHMAN CALCULUS

R. M. REDHEFFER, University of California, Los Angeles

The object of this note is to show that certain theorems in stability theory can be given a simple, unified approach. The main tools used are Rolle's theorem, together with the familiar formula for differentiating an exponential. The average proof-length is about two lines.

A prime denotes differentiation with respect to t, the various functions are assumed continuous, $t \geqq 0$, and

$$A(t) = \int_0^t a(s)ds, \qquad E(t) = e^{-A(t)}, \qquad \bar{\epsilon}(t) = \int_0^t E(s)\epsilon(s)ds.$$

One-sided inequalities. We begin with the following observation:

REMARK 1. *Let $W' \leqq 0$ at points where $W > 0$, and let $W(0) \leqq 0$. Then $W \leqq 0$.*

Indeed, if $W(t_1) > 0$ then there is a t_0, with $0 \leqq t_0 < t_1$, such that $W(t_0) = 0$ and $W > 0$ on (t_0, t_1). But the mean-value theorem gives a contradiction,

$$W(t_1) = W(t_1) - W(t_0) = (t_1 - t_0)W'(t) \leqq 0.$$

REMARK 2. *Let $w' - a(t)w \leqq 0$ at points where $w > 0$, and let $w(0) \leqq 0$. Then $w \leqq 0$.*

The hypothesis implies that $(Ew)' = E(w' - aw) \leqq 0$ at points where $Ew > 0$. Applying Remark 1 to the function $W = Ew$ we get $W \leqq 0$, which is to say, $w \leqq 0$.

The choice $w = u - v$ in Remark 2 gives:

REMARK 3. *Let $Tu = u' - a(t)u$, let $u(0) \leqq v(0)$, and let $Tu \leqq Tv$ hold at all points where $u > v$. Then $u \leqq v$.*

The equation $Tv = \epsilon(t)$ can be written $(Ev)' = E\epsilon$, hence solved by inspection. The solution with $v(0) = \delta$ satisfies $Ev = \delta + \bar{\epsilon}$. Using this v, and writing w instead of u in Remark 3, we get:

REMARK 4. *If $w' - a(t)w \leqq \epsilon(t)$ and $w(0) \leqq \delta$ then $w \leqq (\delta + \bar{\epsilon})e^{A(t)}$, the differential inequality being needed only at points where w exceeds the desired upper bound, v.*

Upon applying this result to $w = u - v$, the reader will get a statement that is related to Remark 4 just as Remark 3 is related to Remark 2. Another application is:

REMARK 5. *Let u satisfy the integral inequality*

$$u(t) \leqq \delta + \int_0^t [a(s)u(s) + \epsilon(s)]ds$$

and let $a(t) \geqq 0$. Then $u \leqq (\delta + \bar{\epsilon})e^{A(t)}$.

Call the right-hand side w, so that the given inequality says: $u \leqq w$. Since $a \geqq 0$ we have $w' = au + \epsilon \leqq aw + \epsilon$. Hence the estimate of Remark 4 applies to w, and *a fortiori* to u.

Two-sided inequalities. The fact that our hypothesis is needed only at points where w or $u - v$ exceeds the desired upper bound is now exploited more fully.

REMARK 6. *If $\delta + \bar{\epsilon} \geqq 0$ then*

$$|w'| \leqq a(t)|w| + \epsilon(t) \quad \text{and} \quad |w(0)| \leqq \delta \text{ imply that } |w| \leqq (\delta + \bar{\epsilon})e^{A(t)}.$$

Indeed, the hypothesis implies that $w' \leqq a(t)w + \epsilon(t)$ at points where $w > 0$ and, hence, at points where w exceeds the desired upper bound. Therefore the estimate of Remark 4 holds. Since the hypothesis also applies to $-w$, that completes the proof.

REMARK 7. *Let $Tu = u' - f(t, u)$ and suppose that f admits the estimate*

$$\frac{f(t, u) - f(t, v)}{u - v} \leqq a(t)$$

whenever $u \neq v$. Then $|Tu - Tv| \leqq \epsilon(t)$ and $|u(0) - v(0)| \leqq \delta \Rightarrow |u - v| \leqq (\delta + \bar{\epsilon})e^{A(t)}$.

Define $w = u - v$ and note that the condition $Tu - Tv \leqq \epsilon$ gives

$$w' \leqq \epsilon + \frac{f(t, u) - f(t, v)}{u - v}(u - v) \leqq \epsilon(t) + a(t)w$$

whenever $w > 0$. In particular, this inequality holds at points where w exceeds the desired upper bound. Hence, Remark 4 applies. Since the hypothesis is symmetric in u and v the same estimate holds for $w = v - u$, and the result follows.

The fact that the inequality for f does not involve $|f(t, u) - f(t, v)|$ allows use of negative $a(t)$. It also allows a broader class of functions f than would otherwise be permitted. For example, the statement "$g(x, u\uparrow)$ is monotone" means that

$$s[g(t, u + s) - g(t, u)] \geqq 0,$$

where $u = u(t)$ but s is an independent variable. Letting $s = v - u$ gives the condition of Remark 7 with $a = 0$, $f = -g$, and we conclude:

REMARK 8. *Let $Tu = u' + g(t, u)$ where, at each fixed t, either $g(t, u\uparrow)$ or $g(t, v\uparrow)$ is monotone. Then*

$$|Tu - Tv| \leqq \epsilon(t) \quad \text{and} \quad |u(0) - v(0)| \leqq \delta \Rightarrow |u - v| \leqq \delta + \int_0^t \epsilon(s)ds.$$

The choice $v=0$ in Remark 8 gives the following result (which also follows from Remark 1):

REMARK 9. *Let $Tu=u'+g(t, u)$ where $sg(t, s)\geqq 0$ for all s. Then*

$$|Tu| \leqq \epsilon(t) \quad \text{and} \quad |u(0)| \leqq \delta \Rightarrow |u| \leqq \delta + \int_0^t \epsilon(s)ds.$$

Remarks 7–9 are special cases of:

REMARK 10. *Let $Tu=u'-f(t, u)+g(t, u)$ where f is as in Remark 7 and g as in Remark 8. Then the conclusion of Remark 7 holds.*

Indeed, $f-g$ satisfies the criterion of Remark 7, and $Tu=u'-(f-g)$.

Uniqueness and stability. If u and v are both exact solutions of the problem $Tw=\tau(t)$, $w(0)=\rho$, then $Tu=Tv$, $u(0)=v(0)$, and the foregoing hypotheses hold with $\epsilon=\delta=0$. The conclusion $|u-v|\leqq 0$ implies uniqueness, $u=v$.

A solution u is said to be *stable* if for any $\eta>0$ there is a $\delta>0$ such that

$$|Tu - Tv| \leqq \delta \quad \text{and} \quad |u(0) - v(0)| \leqq \delta \Rightarrow |u - v| \leqq \eta.$$

Evidently, stability implies uniqueness. By inspection of the error bound in Remark 7 we get:

REMARK 11. *On a finite interval $0\leqq t\leqq t_0$ let u be a solution of*

$$u' - f(t, u) = \tau(t), \qquad u(0) = \rho.$$

Suppose that f satisfies the condition of Remark 7 for this u and for all differentiable v. Then the solution u is stable.

For an infinite interval the stability depends on the behavior of the integrals defining $A(t)$ and $\bar{\epsilon}$. The reader can easily formulate various criteria; as an illustration, we have:

REMARK 12. *Let T be as in Remark 7, with $a(t)\leqq 0$ and $\int_0^\infty |a(s)|ds=\infty$. Then*

$$|Tu - Tv| \leqq \epsilon(t) \Rightarrow \limsup_{t\to\infty} |u - v| \leqq \limsup_{t\to\infty} \frac{\epsilon(t)}{|a(t)|}.$$

For proof choose $c>\limsup \epsilon/|a|$, so that $\epsilon\leqq c|a|=-ca$ for $t\geqq t_0$, say. Then

$$\bar{\epsilon} \leqq \int_0^{t_0} e^{-A(s)}\epsilon(s)ds + c\int_{t_0}^t e^{-A(s)}[-a(s)]ds, \qquad t \geqq t_0.$$

Since $-a=(-A)'$ the second integral equals $e^{-A(t)}-e^{-A(t_0)}$. Denoting the first integral by $m=m(t_0)$ we get

$$e^{A(t)}(\delta + \bar{\epsilon}) \leqq e^{A(t)}(\delta + m) + c,$$

the upper limit is $\leqq c$, and the result follows.

If h and $\tilde{h}$ are functions of t, the statement $h=o(\tilde{h})$ means $\lim(h/\tilde{h})=0$ as $t\to\infty$. A special case of Remark 12 yields:

REMARK 13. *Let T be as in Remark 7, with $a(t) \leqq 0$ and $\int_0^\infty |a(s)| ds = \infty$. Then*

$$|Tu - Tv| = o(a) \Rightarrow |u - v| = o(1).$$

Systems of equations. Peeping into the sophomore year, we finally establish:

REMARK 14. *Let $Tu = u' - F(t, u)$, where u and F are vector-valued functions. Suppose that*

$$|F(t, u) - F(t, v)| \leqq a(t)|u - v|.$$

Then $|Tu - Tv| \leqq \epsilon(t)$ and $|u(0) - v(0)| \leqq \delta \Rightarrow |u - v| \leqq e^{A(t)}(\delta + \bar{\epsilon})$.

For proof define $\tilde{w} = u - v$, $w = |\tilde{w}|$, $\Delta F = F(t, u) - F(t, v)$. Then

$$|Tu - Tv| \leqq \epsilon \Rightarrow |\tilde{w}' - \Delta F| \leqq \epsilon \Rightarrow |\tilde{w}'| \leqq \epsilon + |\Delta F| \Rightarrow |\tilde{w}'| \leqq \epsilon + a(t)w.$$

Furthermore $|w(t) - w(t_1)| \leqq |\tilde{w}(t) - \tilde{w}(t_1)|$, hence $|w'| \leqq |\tilde{w}'|$; and the conclusion follows from Remark 6.

Historical note. The history of these theorems covers a span of seventy years and includes the names of many eminent mathematicians. If ϵ and a are chosen to be constant, Remark 6 yields the theorem of Peano in [6]. The same choice with $\delta = 0$ in Remark 5 sharpens the lemma of Gronwall in [5] since, by the mean-value theorem, $e^x - e^0 < xe^x$ for $x > 0$. The choice $\epsilon = 0$ in Remark 5 sharpens the "fundamental theorem of stability theory" of Bellman (see [1] and [2]), in that we do not require $\delta > 0$ or $u \geqq 0$. The choice $\epsilon = \delta = 0$ in Remark 5 gives a theorem of Weyl (see [8], where he proved it by writing the integral as a Stieltjes integral and using mathematical induction). If ϵ and a are constant, Remark 5 gives a result of Faedo in [3], whereas the choice $\epsilon = \delta = 0$ in Remark 7 yields the essence of Giuliano's theorem in [4]. A special case of Remark 3 is stated in [1], but the proof is incomplete. The line of thought leading to Remark 14 is suggested by the work of Walter [7]. I have not happened to come across Remarks 12 and 13, because most authors base their analysis on Remark 5, which requires $a(t) \geqq 0$.

References

1. E. F. Beckenbach and R. Bellman, Inequalities, Springer, 1961, 133–135.
2. R. Bellman, The stability of solutions of linear differential equations, Duke Math. J., 10 (1943) 643–647.
3. S. Faedo, Su un teorema di esistenza di calcolo delle variazioni e una proposizione generale di calcolo funzionale, Ann. Scuola Norm Sup. Pisa, (2) 12 (1943, pub. 1947) 119–133.
4. L. Giuliano, Generalizzazione di un lemma di Gronwall e di una diseguaglianza di Peano, Rend. Accad. Dei Lincei, Cl. di Sci. Fis. Mat. e Nat., (8) 1 (1946) 1264–1271.
5. T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. of Math., (2) 20 (1919) 292–296.
6. G. Peano, Sur le théorème général relatif à l'existence des intégrales des équations différentielles ordinaires, Nouv. Ann. de Math., 11 (1892) 79–82.
7. W. Walter, Eindeutigkeitssätze für gewöhnliche, parabolische und hyperbolische Differentialgleichungen, Math. Z., 74 (1960) 191–208.
8. H. Weyl, Concerning a classical problem in the theory of singular points of ordinary differential equations, Rev. Ci. Lima, 46 (1944) 73–112.

TANGENTS AND DIFFERENTIALS

HUGH A. THURSTON, University of British Columbia

1. A recent paper [1] makes the point that the usual elementary definition of *differential* is inadequate. The paper [2] gives a valid elementary definition of *tangent-line*. The two concepts are related: given a valid definition of tangent to a plane curve, $dx:dy$ can be defined as the direction-ratio of the tangent. This is a particularly good definition of differential for an elementary course; it is both easy to grasp and potentially rigorous (because the treatment of tangents can be made as rigorous as desired).

The definitions are given in Section 2, and the familiar formulae for $dx:dy$ in the explicit, parametric, and implicit cases follow as in Section 3. It turns out (Section 4) that the formula in the implicit case holds more generally than might be expected: it can be proved without using the implicit-function theorem (which requires continuous derivatives).

In a sense, the three problems of defining tangent, defining differential, and differentiating a function-of-a-function are equivalent. We have indicated that the first two are mutually equivalent; and it is well known that, once differentials are defined, differentiation of a function-of-a-function is trivial. Conversely, if the theorem about differentiating a function-of-a-function is known, it could be made the basis of a definition of differential (not quite as general as ours—in fact, bearing the same relation to ours as parametric tangent bears to geometric tangent: for these terms see [2]). This we show in Section 5.

Finally (Section 6) we point out that if we use two different forms of the *same* relation to calculate differentials, then the definition in this paper ensures automatically that we get the same result from each, whereas other definitions do not have this desirable property.

2. DEFINITION 1. *The line L through the point P of the point-set S is a tangent to S at P if P is a limit-point of S and if, given any cone with vertex P and axis L, the line PQ is inside the cone for every point Q of S near enough to P.*

It is clear that a given set has at most one tangent at a given point, and that all tangents to a plane set lie in the plane of the set.

DEFINITION 2. *Given a relation between two variables, say x and y, we let S be the set of points whose coordinates satisfy the relation. We define a binary function whose domain is a subset of S and whose values are ratios as follows: at any point at which S has a tangent, the value of the function is the direction-ratio of the tangent. The value of the function at (x, y) is traditionally denoted by $dx:dy$; and dx and dy are called the differentials of x and y with respect to the given relation.*

A statement such as "The differentials of x and y with respect to the relation $y=x^2$ satisfy the equation $dy=2x\cdot dx$" is traditionally stated as "If $y=x^2$, then $dy=2x\cdot dx$." (Of course, $dy=2x\cdot dx$ means neither more nor less than $dx:dy$

$=1{:}2x$. Indeed, the only statements that can meaningfully be made about dx and dy are statements about their mutual ratio: the statements "$dx=2$" and "$dx=(dy)^2$" mean nothing.)

3. We have at once the following results.

THEOREM 1. *If $F'(a)$ exists, then the value of $dx{:}dy$ at $[a, F(a)]$ with respect to the relation $y=F(x)$ is $1{:}F'(a)$.*

THEOREM 2. *If $x=X(t)$, $y=Y(t)$, $t\in I$ traces a simple arc, if $a\in I$, and if the ratio $X'(a){:}\,Y'(a)$ exists: then the value of $dx{:}dy$ at $[X(a), Y(a)]$ with respect to the relation $x=X(t)$, $y=Y(t)$, $t\in I$, is $X'(a){:}\,Y'(a)$.*

THEOREM 3. *If the ratio $\psi_1(a, b){:}\psi_2(a, b)$ exists and if $\psi(a, b)=0$ and if ψ is differentiable at (a, b), then the value of $dx{:}dy$ at (a, b) with respect to the relation $\psi(x, y)=0$ is*

$$-\psi_2(a, b){:}\ \psi_1(a, b). \tag{i}$$

(Here ψ_1 and ψ_2 are the two partial derivatives of ψ.)

Proofs. Theorems 1 and 2 follow immediately from [2]. For Theorem 3, we note that, because ψ is differentiable at (a, b),

$$[\psi(a + h, b + k) - \psi(a, b) - h\cdot\psi_1(a, b) - k\cdot\psi_2(a, b)]/(h^2 + k^2)^{1/2} \to 0$$

as $(h, k)\to(0, 0)$. Then if $(a+h, b+k)$ is on the graph of $\psi(x, y)=0$, we have

$$(h\cdot\ell + k\cdot m)/(h^2 + k^2)^{1/2} \to 0 \quad \text{as} \quad (h, k) \to (0, 0),$$

where $\ell=\psi_1(a, b)$ and $m=\psi_2(a, b)$.

Now if u is the inclination of the line joining (a, b) to $(a+h, b+k)$ we have

$$\cos u{:}\ \sin u = h{:}k;$$

and if v is the inclination of a line with direction-ratio (i) we have

$$\cos v{:}\ \sin v = -\ m{:}\ell.$$

Therefore

$$|\sin (u - v)| = |\ell\cdot h + m\cdot k|/(h^2 + k^2)^{1/2}\cdot(\ell^2 + m^2)^{1/2} \to 0$$

as $(h, k)\to(0, 0)$. It follows easily that the line through (a, b) with direction-ratio (i) is the tangent there to the curve.

4. Note. If, in Theorem 3, we were to assume continuity of ψ_1 and ψ_2 in a neighbourhood of (a, b), instead of mere differentiability of ψ at (a, b), then the theorem would follow as an easy corollary to the implicit-function theorem. In this connection, it is interesting to notice that differentiability is not enough for the implicit-function theorem.

Specifically: if

$$\phi(x, y) = \begin{cases} x - \frac{1}{2}y - y^2 \cdot \sin y^{-1} & \text{whenever} \quad y \neq 0 \\ x & \text{whenever} \quad y = 0 \end{cases}$$

then $\phi(0, 0)=0$, $\phi_2(0, 0)=-\frac{1}{2}\neq 0$, ϕ is differentiable at $(0, 0)$, but the equation $\phi(x, y)=0$ is not solvable for y at $(0, 0)$.

Theorem 3 as quoted above, then, is stronger than the version obtained from the implicit-function theorem. However, we cannot remove the differentiability proviso from Theorem 3 and rely only on the existence of the two partial derivatives (not both zero). Specifically: if

$$\phi(x, y) = \begin{cases} (8x^3 - y^3)/(x^2 + y^2) & \text{whenever} \quad (x, y) \neq (0, 0) \\ 0 & \text{when} \quad (x, y) = (0, 0), \end{cases}$$

then $\phi(0, 0)=0$, $\phi_1(0, 0)=8$, $\phi_2(0, 0)=-1$; but the line through $(0, 0)$ with direction-ratio $1:8$ is not the tangent there to the curve $\phi(x, y)=0$. (In fact, the tangent has direction-ratio $1:2$.)

5. Once we have a valid definition of differential, we have an immediate proof of the formula for differentiating a function of a function: if $z=G[F(x)]$ we put $F(x)=y$, whence $z=G(y)$. Then $dz=G'(y)\cdot dy$ and $dy=F'(x)\cdot dx$, whence

$$dz = G'[F(x)] \cdot F'(x) \cdot dx.$$

(At present, this proof seems to be confined to nonrigorous treatments such as [**3**], presumably because of the lack of rigorous definitions of tangent and differential.)

Indeed, the function-of-a-function rule and the validity of the differential are, in a sense, equivalent. If the function-of-a-function rule were proved independently (as, indeed, in most treatments it is) then it could be made the basis of an alternative (slightly less general) definition of differential, as follows. We define $dx:dy$ for a curve in parametric form $x=X(t)$, $y=Y(t)$. For the definition to be valid, this ratio must be proved independent of the choice of parameter for the given curve. The crux of the proof turns out to be the function-of-a-function rule. The details are as follows.

LEMMA. *If a simple arc S has parametrizations*

$$x = X(t), \qquad y = Y(t), \qquad t \in I$$

and

$$x = A(t), \qquad y = B(t), \qquad t \in J$$

then there is a continuous function F with inverse G such that

$$X(t) = A[F(t)], \qquad Y(t) = B[F(t)] \qquad \textit{whenever } t \in I,$$

and

$$A(t) = X[G(t)], \qquad B(t) = Y[G(t)] \qquad \textit{whenever } t \in J.$$

Moreover, if $A'[F(c)] \neq 0$, *then* $F'(c)$ *exists.*

Proof. All results, except possibly that expressed by the last sentence, are well known. To prove the last result, we let $a = F(c)$ and define a function H by

$$\begin{cases} H(t) = \dfrac{A(t) - A(a)}{t - a} & \text{whenever } t \in J \textit{ and } t \neq a \\ H(a) = A'(a). \end{cases}$$

Then $\lim_{t \to a} H(t) = A'(a)$, and H is continuous. Also

$$\frac{X(t) - X(c)}{t - c} = \frac{A[F(t)] - A[F(c)]}{t - c}$$

$$= H[F(t)] \cdot \frac{F(t) - F(c)}{t - c} \qquad \text{whenever } t \in I \text{ and } t \neq c.$$

Now

$$\lim_{t \to c} H[F(t)] = \lim_{t \to a} H(t), \text{ because } F \text{ is continuous}$$

$$= A'(a), \text{ as already proved}$$

$$\neq 0.$$

Therefore, for every t in some neighbourhood of c, $H[F(t)] \neq 0$, and so

$$\frac{F(t) - F(c)}{t - c} = \frac{X(t) - X(c)}{t - c} \cdot \frac{1}{H[F(t)]}.$$

Therefore $F'(c)$ exists (and equals $X'(c)/A'(a)$).

Note. If C is an end-point, then the various limits, neighbourhoods, etc., are one-sided, but the proof is otherwise unaltered.

THEOREM. *If the simple arc S has the parametrizations cited in the lemma, if a point has parameters c and d respectively, and if the ratios* $X'(c) : Y'(c)$ *and* $A'(d) : B'(d)$ *exist, then they are equal.*

Proof. The functions F and G of the lemma exist, and clearly $F(c) = d$. Hence, if $A'(d) \neq 0$, then $F'(c)$ exists. Then $X'(c) = A'(d) \cdot F'(c)$ and $Y'(c) = B'(d) \cdot F'(c)$.

Then $F'(c) \neq 0$ (for otherwise $X'(c)$ and $Y'(c)$ would both be zero and so their ratio would fail to exist) and so

$$X'(c) : Y'(c) = A'(d) : B'(d).$$

If, however, $A'(d) = 0$, then $B'(d) \neq 0$ and a similar proof holds.

6. Sometimes a relation can appear in various different forms. For example, the relations

(i) $y=x^{2/3}$ (ii) $x=y^{3/2}$ (iii) $x=t^3$, $y=t^2$ (iv) $x^2-y^3=0$,

are the same: every (x, y) belonging to any of them belongs to all of them.

From our definition, it follows that the differential-ratio with respect to a given relation does not depend on the form in which the relation is expressed. The traditional definition does not have this property; it can be checked in any particular case, but there is no general theorem. (The existence of a vague feeling that something of this kind is needed is shown by the inclusion in many texts of a "consistency theorem" to the effect that if dy is calculated in terms of dx from the relations $y=F(t)$, $t=G(x)$, and also from the relation $y=F[G(x)]$, then the results are the same. This covers only very special cases, and does not suffice to show consistency for any pair of the equations above. It would, however, show consistency for $y=t^2$, $t=x^{1/3}$ and (i).)

To turn for a moment to physics: Boyle's law can be written $P\cdot V=k$ or $P=k/V$ or $V=k/P$, and a physicist would unhesitatingly use any of these forms to obtain the (isothermal) differentials dP and dV, and expect (without need for checking) that they would give the same result. The physicist would be right, and any treatment of differentials which does not yield this property is inadequate for applications.

Here, then, is a point in which the present definition is superior not only to the traditional definition but to the definition in [1].

References

1. H. A. Thurston, The definition of $dy{:}dx$, this MONTHLY, 70 (1963) 539–541.
2. ———, On the definition of a tangent-line, to appear in this MONTHLY.
3. Sylvanus P. Thompson, Calculus made easy, London, 1910.

THE DENSITY OF PYTHAGOREAN RATIONALS

L. H. LANGE AND D. E. THORO, San Jose State College

If a, b, c are positive integers which satisfy $a^2+b^2=c^2$, we call the number a/b a Pythagorean rational. We give here two proofs of the following

THEOREM. *The set of all Pythagorean rationals is dense in the set of all nonnegative real numbers.*

We use the fact that if x and y are positive integers, then

$$(x^2-y^2)^2+(2xy)^2=(x^2+y^2)^2,$$

and hence, with $x>y$, $(x^2-y^2)/2xy$ is a Pythagorean rational.

Proof 1. Let α, β be any prescribed real numbers which satisfy $0\leqq\alpha<\beta<\infty$. We seek positive integers x and y, $x>y$, such that $\alpha<(x^2-y^2)/2xy<\beta$. Letting $t=x/y$, this is equivalent to the search for a rational t which satisfies $\alpha<\frac{1}{2}(t-t^{-1})<\beta$. If we let $g(t)=\frac{1}{2}(t-t^{-1})$ for all positive t, we have $g(1)=0$,

$\lim_{t\to\infty} g(t)=\infty$, and $g'(t)=\frac{1}{2}(1+t^{-2})>0$. Thus, the function g is strictly increasing. Hence there exist unique t_1 and t_2 such that $g(t_1)=\alpha$, $g(t_2)=\beta$, and $1\leqq t_1<t_2$. The choice of any rational t such that $t_1<t<t_2$ then yields a Pythagorean rational, $g(t)$, which satisfies $\alpha<g(t)<\beta$. Q.E.D.

In our second proof we use the density of the rationals once more, yet in a different way, and employ an equivalent definition of density.

Proof 2. Let $\lambda\geqq 0$ be specified. We seek a sequence $\{(x_i^2-y_i^2)/(2x_iy_i)\}$, of Pythagorean rationals such that

$$\frac{x_i^2-y_i^2}{2x_iy_i}\to\lambda.$$

Now, there exists a sequence of distinct positive rationals $\{e_i\}$, $e_i>1$, such that $e_i\to\lambda+\sqrt{(\lambda^2+1)}$. Consequently,

$$\frac{1}{2}\left(e_i-\frac{1}{e_i}\right)\to\lambda.$$

Setting $e_i=x_i/y_i$, with $x_i>0$, $y_i>0$, we have

$$\frac{x_i^2-y_i^2}{2x_iy_i}\to\lambda.$$

Various questions concerning the distribution of these Pythagorean rationals and certain interesting special sequences of such numbers will be considered in a lengthier paper to be published elsewhere.

PROOF OF A FUNDAMENTAL THEOREM ON SEQUENCES

HOWARD E. BELL, Harpur College

The following proof of a well-known theorem may be of interest to teachers of beginning analysis courses.

THEOREM. *Every sequence of real numbers has a monotone subsequence.*

Let $\langle a_n\rangle$ be a sequence of real numbers. Define

$$\begin{aligned} A &= \{i \mid a_i \leqq a_j \text{ for all except finitely many } j\},\\ B &= \{i \mid a_i \geqq a_j \text{ for all except finitely many } j\},\\ C &= I-(A\cup B), \end{aligned}$$

where I represents the set of all positive integers. At least one of these sets must be infinite.

If A is infinite, then for each $i\in A$, $\exists$ a j in A, $j>i$, for which $a_i\leqq a_j$; and we can define a monotone nondecreasing subsequence $\langle a_{n_k}\rangle$ of $\langle a_n\rangle$ by choosing the indices as follows:

$$n_1 = \text{the smallest integer in } A,$$

$$n_{k+1} = \text{the smallest integer in } A \text{ such that } n_{k+1} > n_k, \text{ and}$$

$$a_{n_k} \leqq a_{n_{k+1}}, \qquad k = 1, 2, \cdots.$$

If B is infinite, we construct a monotone nonincreasing subsequence in an analogous way.

In the event that both A and B are finite, there exist for each $i \in C$ integers j, $k \in C$, $j > i$, $k > i$, such that $a_i < a_j$ and $a_i > a_k$. By employing the previous constructions, we can obtain both a monotone increasing subsequence and a monotone decreasing subsequence.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

THE MATHEMATICAL TRIPOS AND MATHEMATICAL EDUCATION IN GREAT BRITAIN

DANIEL PEDOE, Purdue University

I must begin by explaining that the term Tripos **is** the name given to the mathematical and other honours examinations held every year in the University of Cambridge. The Mathematical Tripos was the first Honours examination instituted by that University. This was in the 18th century. The term Tripos originated in the three-legged stool, or tripod, which candidates sat on when they had to prove their merit by disputation, or wrangling, before the advent of written examinations. The term Wrangler is still preserved for those who obtain honours in the Mathematical Tripos. Although examinations in other subjects are also called Triposes, nobody but a mathematician is ever called a wrangler. This makes one wonder how the old examinations in mathematics were conducted! The term Senior Wrangler was reserved for the candidate who came first in the Mathematical Tripos. Until 1910, when Wranglers were no longer listed in order, the title of Senior Wrangler was much coveted, and the list of Senior Wranglers includes many who subsequently did great work in mathematics, such as Stokes, Cayley and the astronomer John Couch Adams, if we restrict ourselves to the 19th century. Of course, many who subsequently became great did not attain to the Senior Wranglership, but came lower down the list. I need only mention James Clerk Maxwell, who was the second Wrangler in 1854.

mission on the Teaching of Science, of which he is president). The symposium discussed "The Coordination of the Teaching of Mathematics and the Teaching of Science" and was attended by about 30 scientists from all fields and all parts of the world.

In October–November he was visiting professor in the Department of Mathematics, Middle East Technical University, Ankara, Turkey. In December he held the Madras appointment. In February and early March he functioned as visiting lecturer to the University of Pakistan, being sponsored by the International Mathematical Union and the Pakistan Academy of Sciences. He gave a general lecture also to the Pakistan Science Congress in Lyallpur in March.

News Release from the University of Chicago

COLLEGE ATTENDANCE BY MU ALPHA THETA MEMBERS

In January 600 questionnaires were mailed to all fifty states in a spot-check of 1963 graduates who were members of MU ALPHA THETA. These students maintained a B average in high school and showed high ability in mathematics.

The national office of MU ALPHA THETA reports that the 528 replies received show that all but 4 are now enrolled in some college or university. Most students replied that they were enjoying college; only 6 said they were not particularly enjoying their college work. The 4 students who were not in college and had no plans to enter college were all girls. One is married; the other three are employed.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; A. E. LIVINGSTON, University of Alberta; L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; H. EVES, University of Maine; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solution (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math., University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before September 30, 1964.

E 1701. *Proposed by R. F. Jackson, University of Toledo*

Prove that for any three points on a parabola with vertical axis,

$$m_1 = m_{12} + m_{13} - m_{23},$$

where m_1 is the slope of the tangent at the first point and m_{ij} is the slope of the chord through the corresponding pair of points. (This is an occasionally useful formula for numerical differentiation with irregularly spaced points.)

E 1702. *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that an integer p divides $\sum_{j=1}^{p-3} j(j!)$ if and only if p is prime.

E 1703. *Proposed by Gyárfás András, Eötvös University, Budapest, Hungary*

A circle of radius R is completely covered with strips of scotch tape of various widths. Prove that the sum of the widths of the strips is not smaller than $2R$.

E 1704. *Proposed by Stephen Hoffman, Trinity College, and R. B. Killgrove, San Diego State College*

If R is an $m\times n$ rectangle formed from unit squares, find the number of squares containing a segment of one diagonal of R.

E 1705. *Proposed by David and Geralda Singmaster, Berkeley, California*

Given a triangle with sides a, b, c, with $a\leqq b\leqq c$. Define the *skewness* of the triangle to be

$$S = \max(a/b,\ b/c,\ c/a)\min(a/b,\ b/c,\ c/a).$$

Find the maximum and minimum skewness of a triangle. What triangles achieve the maximum and minimum?

E 1706. *Proposed by Michael Fried, University of Michigan*

The *convex subarea* of a convex n-gon is the area in the plane in which a point may be placed such that this point plus the vertices of the n-gon form a convex $(n+1)$-gon. If S represents the perimeter of a convex n-gon, what is the minimum possible value for its convex subarea?

E 1707. *Proposed by D. P. Roselle, Duke University*

Prove that the nth order determinant $|A_{rs}(x)|$, where $A_{rs}(x)=x^{(r-1)(r-s)}$, has the value $\prod_{j=1}^{n-1}(1-x^j)^{n-j}$.

E 1708. *Proposed by J. F. Ramaley, University of California at Berkeley*

Call a matrix M *integral* if and only if the entries of the matrix are all integers. Given n integers $a_1, \cdots, a_n$ show that there exists an $n\times n$ integral matrix M whose first row consists of the n given integers and whose inverse is also integral if and only if g.c.d.$(a_1, \cdots, a_n)=1$.

E 1709. *Proposed by Jack Winter, System Development Corporation, Santa Monica, California*

Show that, for any positive integer n, there exists a sequence of at least n consecutive integers each of which contains a squared factor.

E 1710. *Proposed by D. I. A. Cohen, Princeton University, and Ralph Greenberg, University of Pennsylvania*

If a and b are relatively prime integers, prove that there are infinitely many perfect powers of the form $an+b$.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Adieu to 1963

E 1621 [1963, 890]. *Proposed by Arthur Engel, Stuttgart, Germany*

What is the smallest value of a for which 82^n+a69^n is divisible by 1963 for all odd positive integers n?

I. *Solution by R. J. Herbert, D. T. Kexel, and P. J. Welsh, John Carroll University.* Note that $1963=(151)(13)$. Since $82^n+69^n\equiv 0 \bmod 151$ and $82^n-69^n\equiv 0 \bmod 13$ for all odd n, we must have $a\equiv 1 \bmod 151$ and $a\equiv -1 \bmod 13$. The smallest positive value for a is 454.

II. *Solution by Nyles Barnert, Arcon Corporation, Lexington, Mass.* We employ the lemma: *If x^2-y^2 divides $x+ay$, then it divides $x^{2n+1}+ay^{2n+1}$ for all n.* For the given problem, we set $x=82$, $y=69$, then $x^2-y^2=1963$. We need only find the smallest a such that 1963 divides $82+69a$. This yields $a=454$ as the solution.

Also solved by Gyárfás András, Joseph Arkin, J. W. Baldwin, Merrill Barnebey, Walter Bluger, Adelaide J. Brooks, Brother R. F. Schnepp, Sarvadaman Chowla, B. G. Clark, D. I. A. Cohen, M. J. Cohen, Hüseyin Demir, C. L. Dotton, F. J. Duarte, Philip Franklin, Michael Fried, Anton Glaser, Michael Goldberg, Myron Goldstone, Jerry Goodman, Ralph Greenberg, S. H. Greene, Emil Grosswald, J. H. Halton, R. F. Jackson, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), Frank Kocher, Sidney Kravitz, A. I. Lieberman, N. F. Lindquist, J. J. Malone, Jr., D. C. B. Marsh, Michael Merritt, P. N. Muller, K. A. K. Murthy, Walter Penney, Stanton Philipp, M. Raghavachari, T. S. Ravisankar, Robert Spitz, J. K. Stewart, G. C. Thompson, A. M. Vaidya, Simon Vatriquant, Gary Venter, W. C. Waterhouse, Charles Wexler, Oswald Wyler, Aleksandras Zujus, and the proposer.

Barnert and Venter showed that if nonintegral a are permitted, then $a=1881/69$; Lieberman, Merritt, and Muller showed that if n is even, then $a=1962$. To anticipate similar problems for the next two years, Franklin pointed out that $248^n+(492)243^n$ is divisible by 1964 for all odd n, and $73^n+(1049)58^n$ is divisible by 1965 for all odd n.

Convergence of Two Series

E 1622 [1963, 890]. *Proposed by Michael Gemignani, University of Notre Dame*

Determine for what values of x the following series converge:

$$(1)\qquad \sum_{n=1}^{\infty} (\sin 1/n)^x, \qquad\qquad (2)\qquad \sum_{n=1}^{\infty} (1-\cos 1/n)^x.$$

Solution by Stanton Philipp, Seal Beach, Calif. One can see from the Taylor series of $\sin 1/n$ and $(1-\cos 1/n)$ in powers of $1/n$ that $1/2n<\sin 1/n<1/n$ and

$1/4n^2 < 1 - \cos 1/n < 1/2n^2$. It follows immediately that (1) converges for $\mathrm{Re}(x) > 1$ and (2) converges for $\mathrm{Re}(x) > 1/2$.

Also solved by E. R. Barnes, H. L. Chow, D. I. A. Cohen, M. J. Cohen, Frank Dapkus, J. A. Faucher, Michael Fried, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, Eldon Hansen, H. E. Heatherly, Erwin Just and Norman Schaumberger (jointly), Joel Kugelmass, E. S. Langford, R. D. Leitch, A. I. Lieberman, E. L. Magnuson, D. C. B. Marsh, Morris Morduchow, C. B. A. Peck, L. J. Pratte, George Purdy, Perry Scheinok, C. P. Seguin, D. L. Silverman, R. A. Smith and A. M. Vaidya (jointly), O. E. Stanaitis, Rory Thompson, Andy Vince, Charles Wexler, Raymond Whitney, and Oswald Wyler. Solved partially by Merrill Barnebey, Michael Goldberg, D. E. Myers, W. C. Waterhouse, and the proposer.

The Richness of Mathematical Attack

E 1623 [1963, 891]. *Proposed by R. C. Thompson, University of British Columbia*

Let $f(x)$ be a monic polynomial of degree n with distinct zeros $x_1, x_2, \cdots, x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} g(x_j)/f'(x_j) = 1.$$

I. *Solution by F. R. Olson, State University of New York at Buffalo.* Let

$$f_j(x) = \prod_{i \neq j} (x - x_i) = f(x)/(x - x_j).$$

Then $f'(x_j) = f_j(x_j)$. In terms of Lagrange's interpolation formula

$$g(x) = \sum_{j=1}^{n} g(x_j) f_j(x)/f_j(x_j).$$

Division of the $(n-1)$-th derivative of each side by $(n-1)!$ yields the desired result.

II. *Solution by W. C. Waterhouse, Harvard University.* Expanding in partial fractions we have

$$g(x)/f(x) = \sum_{j=1}^{n} [g(x_j)/f'(x_j)](x - x_j)^{-1}.$$

Now multiply by x and let $x \to \infty$.

III. *Solution by A. E. Danese, State University of New York at Buffalo.* Let $r(z) = g(z)/f(z)$. Then $z = x_1, x_2, \cdots, x_n$ are the only singular points of r and they are simple poles. Hence the sum of the residues of r at these poles is

$$A = \sum_{j=1}^{n} g(x_j)/f'(x_j).$$

The residue of r at $z = \infty$ equals the residue of $-r(1/z)/z^2$ at $z = 0$, which is readily determined as -1. Since the sum of the residues at all the singular points and at the point of infinity is zero, we have that $A=1$.

Also solved by Martin Billik and Eldon Hansen (jointly), J. L. Brown, Jr., Leonard Carlitz, A. J. Chandy, D. I. A. Cohen, M. J. Cohen and Nicholas Derzko (jointly), J. B. Deeds, Hüseyin Demir, Michael Fried, Myron Goldstein, S. H. Greene, W. J. Hartman, J. C. Hickman, V. E. Hoggatt, Jr., R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), A. M. Kriegsman, D. C. B. Marsh, Jim Morrow, M. G. Murdeshwar, C. B. A. Peck, Stanton Philipp, Henry Ricardo, S. M. Robinson, Perry Scheinok, C. P. Seguin, R. F. Shanny, O. E. Stanaitis, E. C. Stopher, V. Vitek, J. E. Wilkins, Jr., A. B. Wilcox, K. S. Williams, Oswald Wyler, David Zeitlin, and the proposer.

This problem was solved by many ingenious attacks. For example, in addition to the above (which were the most commonly employed methods): Hartman, Hickman, and Robinson employed the formula for the $(n-1)$-st divided difference for an arbitrary polynomial of degree $k<n-1$; Murdeshwar used a result in the theory of equations given as Problem 4, p. 172 of vol. 1 of Burnside and Panton, *The Theory of Equations* (5th ed.); Fried, Goldstein, Scheinok, and Wyler used some theory of Vandermonde determinants; Demir employed two geometrical relations of Chasles and Euler involving n distinct fixed points and one arbitrary point of a line.

A Correct and an Incorrect Inequality

E 1624 [1963, 891]. *Proposed by C. M. Frye, San Mateo, California*

Prove, for all integers $n>2$, that $(2n-1)^n+(2n)^n<(2n+1)^n$ and that $(2n)^n+(2n+1)^n>(2n+2)^n$.

Solution by Erwin Just and Norman Schaumberger, Bronx Community College. The first inequality is equivalent to $(2+1/n)^n-(2-1/n)^n>2^n$, which is readily verified for $n>2$ by expanding the left side of the inequality. The second assertion is false; for $\lim_{n\to\infty}[(1+1/n)^n-(1+1/2n)^n]=e-\sqrt{e}>1$, and it follows that, for n sufficiently large, $(1+1/n)^n-(1+1/2n)^n>1$. Multiplying both sides of the latter inequality by $(2n)^n$ we obtain $(2n+2)^n-(2n+1)^n>(2n)^n$, which contradicts the second inequality.

Also solved by A. N. Aheart, Joseph Arkin, J. W. Baldwin, Adelaide J. Brooks, Leonard Carlitz, Allan Chuck, B. G. Clark, M. J. Cohen, Hüseyin Demir, G. C. Dodds, J. A. Faucher, C. E. Franti, Michael Fried, Myron Goldstein, R. B. Grayless, S. H. Greene, Emil Grosswald, J. R. Hanna, Eldon Hansen, Mark Hayamizu, Stephen Hoffman, R. F. Jackson, A. M. Kriegsman, N. F. Lindquist, D. C. B. Marsh and W. H. Laubach (jointly), Stanton Philipp, Arthur Porges, George Purdy, Marlow Sholander, O. E. Stanaitis, G. C. Thompson, Simon Vatriquant, Charles Wexler, Aleksandras Zujus, and the proposer. A number of these solutions were only partially correct.

It can be shown that the second inequality is true if $1\leqq n\leqq 15$, but is false if $n\geqq 16$. In connection with the first inequality, Sholander established the more general result: "Given integer $n>2$ and real numbers x, y, z such that $0<x\leqq y-1\leqq z-2$, then $x^n+y^n\geqq z^n$ implies $x>2n-1$, $y>2n$, $z>2n+1$." Hansen showed that the second inequality should be replaced by $(2n+2)^n-(2n)^n<2(2n+1)^n\sinh(1/2)$.

An Application of the Arithmetic-Geometric Inequality

E 1625 [1963, 891]. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Let n be a positive integer, $\sigma(n)$ the sum of the positive divisors of n, and

$t(n)$ the number of these positive divisors. Show that $\sigma(n)/t(n) \geqq \sqrt{n}$.

Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pa. Consider the divisors in pairs x, y, where $xy=n$. For each pair, $(x+y)/2 \geqq \sqrt{xy} = \sqrt{n}$. Summing corresponding sides of this inequality over all pairs gives $\sigma(n)/2 \geqq [t(n)/2]\sqrt{n}$, or $\sigma(n)/t(n) \geqq \sqrt{n}$.

Also solved by A. N. Aheart, Jeanne A. Baird, J. W. Baldwin, E. R. Barnes, William Becker, D. A. Breault, Leonard Carlitz, Allan Chuck, B. G. Clark, D. I. A. Cohen, D. M. Cohen, Martin Cohen, D. M. Danvers, J. B. Deeds, Hüseyin Demir, Michael Fried, Anton Glaser, David Golber, Jerry Goodman, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), J. C. Lazzara, A. E. Livingston and M. G. Murdeshwar (jointly), C. R. MacCluer, Andrzej Makowski, D. C. B. Marsh, Robert Marsh, Michael Merritt, P. N. Muller, W. I. Nissen, Jr., J. H. Oppenheim, Stanton Philipp, M. Perisastri, A. M. Vaidya, Andy Vince, W. C. Waterhouse, Charles Wexler, Raymond Whitney, K. S. Williams, K. L. Yocom, and the proposer.

An Extension of the Steiner-Lehmus Theorem

E 1626 [1963, 891]. *Proposed by Cornelius Mack, Bradford Institute of Technology, Bradford, England*

Given that X, Y are points on the sides BC, AC of a triangle ABC such that $\measuredangle XAB : \measuredangle CAB = \measuredangle YBA : \measuredangle CBA = \lambda : 1$, where $0<\lambda<1$, show that

(a) $AX>BY$ implies $AC>BC$, and conversely,

(b) $CY>CX$ implies $AC>BC$, and conversely,

(c) $AY>BX$ implies $AC>BC$, and conversely, provided that $0<\lambda\leqq 0.5$, but that there exist triangles for which this is not true if $0.5<\lambda<1$.

Solution by the proposer. (a) Now $\measuredangle AXC=B+\lambda A$, $\measuredangle BYC=A+\lambda B$. Hence $AX \sin(B+\lambda A)=AC\sin C$. Similarly, $BY\sin(A+\lambda B)=CB\sin C$. Hence

$$AX/BY = \sin B \sin(A+\lambda B)/\sin A \sin(B+\lambda A).$$

Consider

$$\alpha = 2\sin B \sin(A+\lambda B) - 2\sin A \sin(B+\lambda A).$$

If we set $1-\lambda=\mu$, then

$$\alpha = \cos(\mu B - A) - \cos(A+B+\lambda B) - \cos(B-\mu A) + \cos(A+B+\lambda A).$$

Collecting the first and third, and the second and fourth terms we get

$$\alpha = 2\sin\lambda\phi\sin(2-\lambda)\theta + 2\sin\lambda\theta\sin(2+\lambda)\phi, \tag{1}$$

where $2\theta=B-A$, $2\phi=B+A$. If $B>A$, then, since $0<B+A<\pi$ and $0<\lambda<1$, we have $0<\lambda\theta<\lambda\phi<\pi/2$; $(2-\lambda)\theta<B-A<\pi$. If, further,

$$(2+\lambda)\phi \equiv (1+\lambda/2)(B+A) < \pi,$$

every term in (1) is positive, and therefore so is α. If $(1+\lambda/2)(B+A)>\pi$, nevertheless $(1+\lambda/2)(B+A)-\pi<\lambda\phi$. Hence $-\sin(2+\lambda)\phi<\sin\lambda\phi$. But

$$\sin (2-\lambda)\theta - \sin \lambda\theta = 2 \sin \mu\theta \cos \theta > 0.$$

Hence $\alpha>0$ in this case also. But if $\alpha>0$, then $AX>BY$. By symmetry, if $A>B$, then $BY>AX$. Hence $B>A$ implies $AX>BY$ and conversely, and $B=A$ implies $AX=BY$ and conversely.

(c) Now $AY/BX=\sin \lambda B \sin (B+\lambda A)/\sin \lambda A \sin (A+\lambda B)$. Consider

$$\begin{aligned}\beta &= 2 \sin \lambda B \sin (B+\lambda A) - 2 \sin \lambda A \sin (A+\lambda B)\\ &= \cos (\mu B+\lambda A) - \cos \{B+\lambda(A+B)\} - \cos (\mu A+\lambda B)\\ &\quad + \cos \{A+\lambda(A+B)\}\\ &= 2 \sin \phi \sin \rho\theta + 2 \sin \theta \sin (2-\rho)\phi = f(\theta, \phi), \text{ say,}\end{aligned}$$

where $\rho\equiv 2\mu-1\equiv 1-2\lambda$. If $0<\lambda\leqq 0.5$, then $\rho>0$ and every term in $f(\theta, \phi)$ is positive if $B>A$. Hence $\beta>0$ and $AY>BX$. Again, if $A>B$, then $BX>AY$, and so if $0<\lambda\leqq 0.5$, $AY>BX$ implies $AC>BC$ and conversely. If $\lambda=0.5+\epsilon$, $\epsilon>0$, then $\mu=0.5-\epsilon$ and $\rho<0$. Since $2\phi\equiv A+B$ can approach π it is possible to choose $A+B$ so that $(2-\rho)\phi>\pi$, in which case both products in $f(\theta, \phi)$ are negative, and so with $B>A$ it is possible to have $AY<BX$ and conversely.

(b) This is surprisingly difficult. We have

$$CX = AC \sin \mu A/\sin (\lambda A+B), \qquad CY = CB \sin \mu B/\sin (\lambda B+A).$$

Hence

$$CY/CX = \sin (\lambda A+B) \sin \mu B \sin A/\sin (\lambda B+A) \sin \mu A \sin B.$$

Consider

$$\begin{aligned}\gamma &\equiv \sin (\lambda A+B) \sin \mu B \sin A - \sin (\lambda B+A) \sin \mu A \sin B\\ &= \sin (A+B)\{\cos \mu A \sin \mu B \sin A - \cos \mu B \sin \mu A \sin B\}\\ &\quad + (\sin B - \sin A) \cos (A+B) \sin \mu A \sin \mu B.\end{aligned}$$

Hence

$$\begin{aligned}\gamma/\cos \phi &= \sin \phi\{(\sin B+\sin A) \sin 2\mu\theta - (\sin B-\sin A) \sin 2\mu\phi\}\\ &\quad + 2 \sin \theta \cos (A+B) \sin \mu A \sin \mu B,\end{aligned}$$

and

$$\begin{aligned}\gamma/(2 \cos \phi) &= \sin^2 \phi\{\sin 2\mu\theta \cos \theta - \cos 2\mu\theta \sin \theta\}\\ &\quad - \sin \phi \sin \theta\{\sin 2\mu\phi \cos \phi - \cos 2\mu\phi \sin \phi\} + \sin \theta \sin \mu A \sin \mu B\\ &= \sin^2 \phi \sin \rho\theta - \sin \phi \sin \theta \sin \rho\phi + \sin \theta \sin \mu A \sin \mu B.\end{aligned}$$

Now, since if $0<\mu\leqq 0.5$, $\rho\equiv 2\mu-1<0$, and if $B>A$,

$$\sin (1-2\mu)\phi/\sin \phi > \sin (1-2\mu)\theta/\sin \theta,$$

since $\sin \omega x/\sin x$ is an increasing function of x for $0<\omega<1$, $0<x<\pi$. Hence

$\gamma/(2\cos\phi)>0$, and therefore $\gamma>0$. Hence $CY>CX$. If $\mu>0.5$, then, since

$$2\sin\mu A\sin\mu B=\cos(\rho+1)\theta-\cos(\rho+1)\phi,$$

we see from above that

$$\begin{aligned}\gamma/\cos\phi &= (2\sin^2\phi-\sin^2\theta)\sin\rho\theta-2\sin\theta\sin\phi\sin\rho\phi+\sin\theta\\ &\quad+\sin\theta\{\cos\rho\theta\cos\theta-\cos\rho\phi\cos\phi\}\\ &=2(\sin^2\phi-\sin^2\theta)\sin\rho\theta+\sin\theta\{\cos(1-\rho)\theta-\cos(1-\rho)\phi\}.\end{aligned}$$

But $0<(1-\rho)\theta<(1-\rho)\phi<\pi/2$. Hence every term in our expression for $\gamma/\cos\phi$ is positive. Hence $\gamma>0$, and therefore $CY>CX$; and so $CY>CX$ implies $AC>BC$ and conversely, while $CY=CX$ implies $AC=BC$.

Editorial Note. Simpler solutions to this problem, particularly to part (b), are invited.

Square-Free Integers

E 1627 [1963, 891]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that every positive integer except 1 is the sum of two square-free integers.

Solution by Sarvadaman Chowla, R. A. Smith, and A. M. Vaidya, Pennsylvania State University. If for a real number x, $Q(x)$ denotes the number of square-free positive integers less than or equal to x, then it is enough to show that $Q(x)>(x+1)/2$; for then, if $x_1,\cdots,x_k$ be the square-free positive integers $\leqq n$, consider the two sets

$$M_1:\{x_1,\cdots,x_k\},\qquad M_2:\{n-x_1,\cdots,n-x_k\}.$$

M_1 contains k distinct positive integers $\leqq n$ and M_2 contains k distinct non-negative integers strictly less than n, that is, M_2 contains at least $k-1$ distinct positive integers strictly less than n. Since $2k-1>n$, we have an $x_i\neq n$ and an x_j such that $x_i=n-x_j$. Then n is the sum of two square-free positive integers x_i and x_j. We shall show that $Q(n)>(n+1)/2$ for $n\geqq 385$. The assertion of the problem can be verified directly for all smaller values of n.

It can easily be shown (see, e.g., Landau's *Primzahlen*, p. 581) that

$$Q(x)=\sum_{n\leqq\sqrt{x}}\mu(n)[x/n^2],$$

where $\mu(n)$ is the Möbius function and $[u]$ denotes as usual the greatest integer $\leqq u$. Therefore

$$\begin{aligned}Q(x)&=x\sum_{n=1}^{\infty}\mu(n)/n^2-x\left\{\sum_{n>\sqrt{x}}\mu(n)/n^2+\sum_{n\leqq\sqrt{x}}\mu(n)(x/n^2-[x/n^2])\right\}\\ &=(6/\pi^2)x-\{S_1+S_2\},\ \text{say}.\end{aligned}$$

Now it can be proved by elementary means that

$$|S_1|\leqq 1+\sqrt{x}\quad\text{and}\quad|S_2|<\sqrt{x}.$$

Hence

$$|Q(x) - (6/\pi^2)x| < 1 + 2\sqrt{x}.$$

Now if $x \geqq 385$, then

$$1 + 2\sqrt{x} \leqq 7x/66 - 1/2.$$

So finally, for $n \geqq 385$,

$$Q(n) > (6/\pi^2 - 7/66)n + 1/2 > (20/33 - 7/66)n + 1/2 = (n+1)/2.$$

Also solved by Gyárfás András, J. W. Baldwin, W. R. Becker, David Bienenfeld, M. J. Cohen, Frank Dapkus, George Diderrich, Michael Fried, S. H. Greene, Emil Grosswald, Ned Harrell, R. A. Jacobson, D. C. B. Marsh, Michael Merritt, Stanton Philipp, G. C. Thompson, Jack Winter, and the proposer.

A number of these solutions were open to criticism.

Vaidya called attention to T. Estermann's paper, "On the representation of a number as the sum of two numbers not divisible by kth powers," in the *J. London Math. Soc.*, 6 (1931) 37–40. It may be of interest to know that Estermann has proved (*ibid*, 219–221) that every large number is the sum of a prime and a square-free integer.

Some Triangle Inequalities Involving the Angle Bisectors

E 1628 [1963, 891]. *Proposed by Leonard Carlitz, Duke University*

Let t_a, t_b, t_c denote the angle bisectors of a triangle, r the inradius, R the circumradius, and s the semiperimeter. Show that

$$t_a^2 + t_b^2 + t_c^2 \leqq s^2, \tag{1}$$

$$t_b^2 t_c^2 + t_c^2 t_a^2 + t_a^2 t_b^2 \leqq rs^2(4R + r), \tag{2}$$

$$t_a t_b t_c \leqq rs^2. \tag{3}$$

In each case there is equality if and only if the triangle is equilateral.

Solution by Stanton Philipp, Seal Beach, Calif. It is easy to prove that $t_a = [2\sqrt{bc}/(b+c)]\sqrt{s(s-a)}$. Then $t_a \leqq \sqrt{s(s-a)}$, with equality if and only if $b=c$. Similar statements hold, of course, for t_b and t_c. Now the assertions to be proved follow immediately, since $bc+ca+ab = s^2+4rR+r^2$, $2s=a+b+c$, $rs^2 = \sqrt{s^3(s-a)(s-b)(s-c)}$.

Also solved by A. N. Aheart, W. J. Blundon, H. W. Guggenheimer, J. S. Leon, Franz Leuenberger, Andrzej Makowski, D. C. B. Marsh, and the proposer.

A Condition for a Semigroup to be an Abelian Group

E 1629 [1963, 891]. *Proposed by F. M. Sioson, University of Hawaii*

Show that any associative system S satisfying the identity $x^2y = y = yx^2$ is a commutative group.

I. *Solution by Roy Dubisch, University of Washington, and B. E. Rhoades,* *Berkeley, Calif.* The given identity implies $x^2=e$ and each x is its own inverse. $(xy)^2=e$ yields $xy=y^{-1}x^{-1}=yx$.

II. *Solution by C. M. Geschke, John Carroll University.* By a well-known theorem any associative system is a group with respect to a binary composition if it contains an identity from the right (r) and for every element y a right inverse (y_r) with respect to r. Now $yx^2=y\to x^2=r$ and $yy=y^2=r\to y=y_r$. Hence S is a group. Also, $xx=r=x(yy)x=(xy)(yx)\to yx=(xy)_r$. But $(xy)_r=xy$. Hence S is commutative. The proof shows that the hypothesis can be weakened by requiring only the identity $yx^2=y$ to be satisfied.

Also solved by A. N. Aheart, Joseph Altinger, W. H. Bailey, K. F. Bailie, Nyles Barnert, Ralph Bennett, D. A. Breault, Brother T. C. Wesselkamper, R. J. Bumcrot, F. B. Cannonito, Leonard Carlitz, A. J. Chandy, D. I. A. Cohen, M. J. Cohen, R. J. Cormier, D. M. Danvers, J. B. Deeds, Hüseyin Demir, George Diderrich, Roy Feinman, M. S. Fineman, T. S. Frank, Michael Fried, J. A. Glasenapp and T. C. Upson (jointly), Anton Glaser, Jack Goebel, Michael Goldberg, Myron Goldstein, D. J. Hansen, Dunstan Hayden, H. E. Heatherly, Stephen Hoffman, J. E. Homer, Jr., W. D. Jackson, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), P. L. Kingston, Max Klicker, Joel Kugelmass, J. Kuzmanovich, E. S. Langford, J. F. Leetch, Joel Levy and P. Meyers (jointly), Jiang Luh, R. J. Lundgren, C. R. MacCluer, J. J. Malone, Jr., D. C. B. Marsh, Stephen Montague, Jim Morrow, M. G. Murdeshwar, D. E. Myers, John Nichols, W. I. Nissen, Jr., C. B. A. Peck, M. Perisastri, Stanton Philipp, D. T. Price, George Purdy, T. S. Ravisankar, P. N. Rheinstein, James Riddell, Azriel Rosenfeld, Perry Scheinok, Marlow Sholander, D. L. Silverman, John Stout, Rory Thompson, A. M. Vaidya, W. C. Waterhouse, Ron Wilder, J. E. Wilkins, Jr., A. B. Wilcox, Oswald Wyler, K. L. Yocom, and the proposer.

A Bounded Solution of a Differential Equation

E 1630 [1963, 891]. *Proposed by Reuben Hersh, Stanford University*

If the polynomial $P(x)$ has no purely imaginary zeros, and if the function f satisfies $|f(x)|<1$ for all real x, then the ordinary differential equation $P(D)u=f$ has exactly one solution $u(x)$ which is bounded for all x, and that bound can be chosen as the product of the reciprocals of the real parts of the zeros of P.

Solution by Oswald Wyler, University of New Mexico. Since no solution of $P(D)u=0$ is bounded for all real x, there is at most one bounded solution of $P(D)u=f$ for bounded f. Denote it (if it exists) by P^*f. If $P=QR$, then $P^*f=Q^*(R^*f)=R^*(Q^*f)$ if Q^* and R^* are defined. Thus it is enough to produce P^*f for $P(x)=x-c$. We put

$$(P^*f)(x) = \int_{-\infty}^{x} e^{c(x-t)}f(t)dt, \qquad \text{if } \operatorname{Re} c < 0;$$

$$(P^*f)(x) = -\int_{x}^{\infty} e^{c(x-t)}f(t)dt, \qquad \text{if } \operatorname{Re} c > 0.$$

One checks easily that $|(P^*f)(x)|\leq K/|\operatorname{Re} c|$ for all real x if $|f(x)|\leq K$ for all real x, and that $D(P^*f)-c(P^*f)=f$, if either $\operatorname{Re} c<0$ or $\operatorname{Re} c>0$.

Also solved by the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before December 31, 1964.

5209. *Proposed by Bezalel Peleg, Hebrew University, Jerusalem*

For each matrix B whose entries are rational numbers, let $m(B)$ be the least common denominator of the entries of B. Show that for each n, there is a nonsingular $n\times n$ matrix A all of whose entries are 0, 1 or -1, such that

$$m(A^{-1}) \geqq \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^{n-1}\sqrt{5}} - 1 \sim 0.9(1.6)^n$$

Hadamard's formula yields $m(A^{-1}) \leqq n^{n/2}$. Can either estimate be improved?

5210. *Proposed by T. J. Kaczynski, Evergreen Park, Illinois*

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying:

(1) K is an abelian group under addition,
(2) $K-\{0\}$ is a group under multiplication, and
(3) $x(y+z)=xy+xz$ for all $x, y, z\in K$.

Suppose that for some n, $0=1+1+\cdots+1$ (n times). Prove that, for all $x\in K$, $(-1)x=-x$.

5211. *Proposed by R. M. Redheffer, University of California, Los Angeles*

A function $u\in C^2$ attains a minimum of value 0 at an interior point of a region B. If the second partial derivatives satisfy $\sum u_{ij}^2\leqq 1$, and if every pair of points of B can be joined by a string of length $\sqrt{2}$ lying wholly in B, then $u<1$ throughout B.

5212. *Proposed by L. Carlitz, Duke University*

Let p be prime and let $\psi(a)=(a/p)$, the Legendre symbol. Show that, if $abcd\not\equiv 0 \pmod{p}$, then

$$S = \sum_{x,y,z=0}^{p-1} \psi(ayz+bzx+cxy+dxyz) = -\,[\psi(-abc)]p.$$

5213. *Proposed by T. I. Seidman, The Boeing Co., Seattle*

Let X be a Banach space, S a closed convex set in X, $x_0\in(X-S)$, $x_1\in S$. Let $d=\inf\{\|x_0-x\|: x\in S\}$ and let $x'\in S$ be such that $\|x_0-x'\|<d+\epsilon$. Prove or disprove that $\|x_1-x'\|\leqq\|x_1-x_0\|+\epsilon$.

5214. *Proposed by E. D. Nix, Norwich, Vermont*

Exhibit, or prove the nonexistence of, an arcwise connected topological space S having more than one point and which is not a homeomorph of any open set of any finite-dimensional Euclidean space, such that

a) homeomorphisms between subsets of S preserve the property of having nonempty interiors.

b) if p and q are any (not necessarily distinct) points of S and if U_p and U_q are any neighborhoods respectively of p and q, then there exists a neighborhood V_q of q such that $V_q \subseteq U_q$ and V_q is homeomorphic to U_p with a homeomorphism $h: U_p \to V_q$ such that $h(p) = q$.

5215. *Proposed by A. Wilansky, Lehigh University*

Prove that a topological group (with more than one element) has the discrete topology if and only if it has a compact open subset which includes no right translate of itself.

5216. *Proposed by Oswald Wyler, University of New Mexico*

Let p be a prime and let F_n be the nth Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Show that:

(a) $F_{p-1} \equiv 0 \pmod p$, $F_p \equiv 1 \pmod p$, if $p \equiv \pm 1 \pmod 5$.

(b) $F_p \equiv -1 \pmod p$, $F_{p+1} \equiv 0 \pmod p$, if $p \equiv \pm 2 \pmod 5$.

5217. *Proposed by Otomar Hajek, Prague, Czechoslovakia*

Given a real-valued function f on a compact interval $J \subset E^1$ of class Lip_A^1 [i.e., $|f(x) - f(y)| \leqq A|x-y|$ for $x, y \in J$], prove that there exist polynomials p_n with $p_n \to f$ uniformly on J, p_n in the same class Lip_A^1 on J.

Using this one may show that for Lip_A^1 maps from a compact parallelepiped of E^p to E^q, there exist uniform polynomial approximations in $\mathrm{Lip}_{A\sqrt{q}}^1$ (in the Euclidean norm). Can this be sharpened to Lip_A^1?

SOLUTIONS OF ADVANCED PROBLEMS

Convergents of a Continued Fraction

5111 [1963, 672]. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If P_n/Q_n is the nth convergent of the continued fraction for $\sqrt{(x^2+1)}$, then $\operatorname{arccot} P_{2n-1} = 2 \operatorname{arccot} Q_{2n} - \operatorname{arccot} P_{2n+1}$.

Solution by D. Suryanarayana, Andhra University, Waltair, India. The proposed continued fraction is

$$x + \frac{1}{2x+}\,\frac{1}{2x+}\cdots.$$

We have the following relations:

$$(1) \qquad P_{2n+1} = 2xP_{2n} + P_{2n-1}, \qquad\qquad (2) \qquad P_{2n}^2 - P_{2n-1}P_{2n+1} = x^2 + 1,$$

$$(3) \qquad P_{2n}^2 - (x^2+1)Q_{2n}^2 = 1, \qquad\qquad (4) \qquad (x^2+1)Q_{2n} = xP_{2n} + P_{2n-1}.$$

[(1) and (2) can be proved by induction on n, and for (3) and (4) see Barnard and Child, *Higher Algebra*, pp. 534–535.] By virtue of the above four relations it is not hard to show that

$$\frac{P_{2n-1}P_{2n+1}-1}{P_{2n-1}+P_{2n+1}}=\frac{Q_{2n}^2-1}{2Q_{2n}}.$$

That is, cot $\{\text{arccot } P_{2n-1}+\text{arccot } P_{2n+1}\}=\cot\{\text{arccot } Q_{2n}\}$ which implies the required result.

Also solved by A. N. Aheart, L. Carlitz, Walter Penney, and J. M. Quoniam.

Non-Archimedean Field

5112 [1963, 672]. *Proposed by N. R. Riesenberg, University of Wisconsin*

In Dieudonné, *Foundations of Modern Analysis* (Academic Press, N. Y., 1960), a real number system is defined as a field which (1) is Archimedean ordered, and (2) possesses the nested interval property. It is well known that neither (1) nor (2) alone suffices to give a real number system and many examples of Archimedean ordered fields which are not real number systems are in the literature. Give an example of a field which is non-Archimedean ordered but which possesses the nested interval property.

Solution by David W. Dean, Duke University. Let F be the field of formal power series over the reals. Say that $\sum_0^\infty a_nz^n \geqq 0$ if $a_n \geqq 0$ for all n. An interval in F is a set of the form $[A, B]=\{C\in F \mid A\leqq C\leqq B\}$.

Suppose $[A_j, B_j]$ is a decreasing sequence of intervals, and that $A_j = \sum_{n=0}^\infty a_n^{(j)}z^n$, $B_j=\sum_{n=0}^\infty b_n^{(j)}z^n$. Then $[a_n^{(j)}, b_n^{(j)}]$ is a decreasing sequence of closed real intervals for each n, and so there exists c_n such that

$$c_n \in \bigcap_{j=1}^{\infty} [a_n^{(j)}, b_n^{(j)}].$$

The series $\sum_{n=0}^\infty c_nz^n$ is then in each $[A_j, B_j]$ and so is in $\bigcap_{j=1}^\infty [A_j, B_j]$.

Finally F is not Archimedean as 1 and z are not related at all.

Also solved by R. O. Davies.

Sum of an Infinite Series

5113 [1963, 672]. *Proposed by J. S. Frame, Michigan State University*

Sum the series

$$S=\sum_{k=0}^{\infty}\binom{2k}{k}(-16)^{-k}(2k+1)^{-2}.$$

Solution by A. Weinmann, The University, Leicester, England. The required sum S can be transformed into a definite integral by using an integral representation for $(2k+1)^{-2}$. Thus

$$S = \sum_{k=0}^{\infty} \binom{2k}{k} (-16)^{-k}(2k+1)^{-2}$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (\tfrac{1}{4})^k \int_0^{\infty} e^{-(2k+1)t} t\,dt$$

$$= \int_0^{\infty} \frac{te^{-t}}{(1+\frac{1}{4}e^{-2t})^{1/2}} dt = 2(A+B-C),$$

on using the substitution $e^{-t} = (z^2-1)/z$ and integrating by parts; here

$$A = \int_1^a \frac{\log z}{(z+1)} dz, \qquad B = \int_1^a \frac{\log z}{(z-1)} dz,$$

$$C = \int_1^a \frac{\log z}{z} dz = \tfrac{1}{2}(\log a)^2,$$

and a is the positive root of $a^2 = a+1$. Let

$$D = \int_0^1 \frac{\log z}{(z+1)} dz = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots = -\pi^2/12.$$

Remembering $a^2 = a+1$ where needed, we have

$$B = \log a \cdot \log(a-1) - \int_1^a \frac{\log(z-1)}{z} dz \qquad \text{(by parts)}$$

$$= -(\log a)^2 - D - \int_1^{a-1} \frac{\log t}{(t+1)} dt \qquad (z-1=t)$$

$$= -2C - D - \int_1^a \frac{\log v}{v(v+1)} dv \qquad (t = 1/v)$$

$$= -2C - D - C + A = -D - 3C + A;$$

$$A + B = \int_1^a \frac{2z\log z}{(z^2-1)} dz = \frac{1}{2}\int_0^a \frac{\log(t+1)}{t} dt \qquad (z^2 = t+1)$$

$$= \tfrac{1}{2}\log(a+1)\cdot\log a - \tfrac{1}{2}D - \tfrac{1}{2}A \qquad \text{(by parts)}$$

$$= 2C - \tfrac{1}{2}D - \tfrac{1}{2}A.$$

Elimination of A and B from these various results gives S in terms of known C and D, so that we obtain finally $S = \pi^2/10$.

Also solved by L. F. Epstein, R. P. Kenan and M. L. Glasser, R. N. Kesarwani, J. Koekoek, Franklin C. Smith, J. H. van Lint, K. W. van Weerden, Jet Wimp, and the proposer.

Function with Uncountable Number of Horizontal Tangents

5114 [1963, 672]. *Proposed by W. E. Johnson and C. M. Petty, Lockheed Aircraft Corp., Sunnyvale, California*

Let the function $F(t)$ have a continuous derivative on $[0, 1]$ and set $S_1=\{t: F'(t)=0\}$, $S_2=\{F(t): t\in S_1\}$. Show by an example that the set S_2 may be uncountable.

Solution by W. C. Waterhouse, Harvard University. Let $g(x)=0$ on the Cantor set and $g(x)=(x-a)(b-x)$ in each interval (a, b) forming the complement of the Cantor set. Let $F(x)=\int_0^x g$; then $F'=g$ is continuous, and S_1 is the uncountable Cantor set. Since $\{x: g(x)>0\}$ is everywhere dense, F is strictly increasing, and hence one-to-one; therefore S_2 is also uncountable.

Also solved by I. N. Baker, Robert Bowen, R. O. Davies, J. L. Denny, R. A. Jacobson, K. F. Kinneberg, K. O. Leland, Solomon Marcus, Ron Rietz, J. M. Shaw and J. F. Standish, D. E. Varberg, Oswald Wyler, Larry Zalcman, J. A. Zilber, and the proposers.

Linear Dimension of Composite Field

5115 [1963, 672]. *Proposed by Harley Flanders, Purdue University*

Let $k\leqq K$, $F\leqq\Omega$, all commutative fields. We may form the composite KF and it is known that $[KF:F]\leqq[K:k]$ if $[K:k]$ is finite. Prove that this inequality is true when $[K:k]$ is infinite, provided that $[F:k]$, the linear dimension of F over k, is countable.

Solution by Oswald Wyler, University of New Mexico. If $[KF:F]$ is finite and $[K:k]$ infinite, then $[KF:F]<[K:k]$ trivially. We assume now that $[KF:F]$ and $[K:k]$ both are infinite, and that $[F:k]$ is countable. In this case, $[KF:F]=[KF:F][F:k]=[KF:k]=[KF:K][K:k]$, and card $K=[K:k]$ card k. If k is countable, then F also is countable, and $[KF:F]=[KF:F]$ card $F=$card $KF=$card $K[F]=$card $K=[K:k]$ card $k=[K:k]$. If k is uncountable and $x\in\Omega$ transcendental over k, then the uncountably many elements $(x-a)^{-1}$ of Ω, $a\in k$, are linearly independent over k. It follows that F is algebraic over k if $[F:k]$ is countable and k uncountable. But then $KF=K[F]$, and since a linear basis of F over k generates the vector space $K[F]$ over K, we have $[KF:K]\leqq[F:k]$. It follows that $[KF:K][K:k]=[K:k]$, so that, again, $[KF:F]=[K:k]$.

We note that our result is somewhat stronger than that proposed in the problem.

Also solved by the proposer.

A Double Summation

5116 [1963, 673]. *Proposed by David Greenstein, Northwestern University*

Let

$$S(A) = \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{A^{j+k}}{j!k!}, \qquad (A \text{ real}).$$

An engineer needs asymptotic information about $S(A)$ as $A \to \infty$. He conjectures that $e^{-2A}S(A) \to 1$. Prove or disprove his conjecture.

Solution by R. G. Buschman, State University of New York at Buffalo. Consider

$$\sum_{n=0}^{N} \sum_{k=0}^{N} \frac{A^{n+k}}{n!k!} = \sum_{n=1}^{N} \sum_{k=0}^{n-1} \frac{A^{n+k}}{n!k!} + \sum_{k=1}^{N} \sum_{n=0}^{k-1} \frac{A^{n+k}}{n!k!} + \sum_{n=0}^{N} \frac{A^{2n}}{n!n!}.$$

If we pass to the limit on N, then we have

$$e^{2A} = 2S(A) + I_0(2A),$$

where I_0 is the modified Bessel function of the first kind. This yields the explicit formula for $S(A)$,

$$S(A) = \tfrac{1}{2}\{e^{2A} - I_0(2A)\},$$

to which the known asymptotic expansion for I_0 can be applied, giving

$$e^{-2A}S(A) = \tfrac{1}{2} + O(A^{-1/2}).$$

Also solved by C. R. Berndtson and C. G. Fain, M. S. Demos, G. Di Antonio, D. Ž. Djoković, Ralph Greenberg, Emil Grosswald, Eldon Hansen, G. W. Hedstrom, J. Koekoek, E. L. Magnuson, Stanton Philipp, D. Ramakotaiah, J. J. Schäffer, Arnold Singer, Franklin C. Smith, R. P. Tapscott, Rory Thompson and Henry Gray, W. F. Trench, J. H. van Lint, W. C. Waterhouse, A. Weinmann, J. Ernest Wilkins, Jr., Oswald Wyler, M. Wyman, and the proposer.

Roots of Unity

5117 [1963, 673]. *Proposed by L. Carlitz, Duke University*

Let η, ζ be roots of unity such that

$$a\eta + b\zeta + c = 0 \qquad (\eta^2 \neq 1, \zeta^2 \neq 1),$$

where a, b, c are nonzero integers. Show that the only possibilities are given by $a=b=c$, $\eta=\omega$, $\zeta=\omega^2$, $\omega^2+\omega+1=0$.

Solution by Harley Flanders, Purdue University. We have $-a\eta=b\zeta+c$ and $-a\bar{\eta}=b\bar{\zeta}+c$. We multiply these expressions, noting that $\bar{\eta}=\eta^{-1}$, $\bar{\zeta}=\zeta^{-1}$ since these are roots of unity:

$$a^2 = b^2 + c^2 + bc(\zeta + \zeta^{-1}).$$

Since $bc \neq 0$ we conclude that $\zeta+\zeta^{-1}=u=$rational, and $\zeta^2-u\zeta+1=0$. Thus ζ is quadratic over the rationals so that if ζ is a primitive nth root of unity, then $n=1, 2, 3, 4, 6$ are the only possibilities; by hypothesis $n\neq 1$, $n\neq 2$. We rule out $n=4$ because if $\zeta=i$, $i^2=-1$, then η is a unity root in the field $Q(i)$ so that $\eta=\pm i$. But $a(\pm i)+bi+c\neq 0$. This leaves two cases:

$$\omega^2+\omega+1=0, \qquad (-\omega)^2+(-1)(-\omega)+1=0,$$

where ω is a primitive cube root of unity so that $-\omega$ is a primitive sixth root. This gives the desired result (with the trivial alternative $a=-b=c$, $\eta=\omega^2$, $\zeta=-\omega$.)

Also solved by W. J. Blundon, S. Chowla and A. M. Vaidya, Martin Cohen, Irving Gerst, Gordon Pall and Olga Taussky Todd, B. Sapolsky, Oswald Wyler, L. Zalcman, and the proposer.

A Nonnegative Trigonometric Polynomial

5118 [1963, 673]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the integer n be given, $n \geqq 2$. Show that if

$$T(t) = 1 + a_1 \cos t + b_1 \sin t + a_n \cos nt + b_n \sin nt \geqq 0 \tag{1}$$

for all real t, then

$$a_1 \leqq \sec(\pi/2n) \tag{2}$$

with equality if and only if

$$T(t) = 1 + \left(\sec\frac{\pi}{2n}\right)\cos t + \frac{(-1)^n}{n}\left(\tan\frac{\pi}{2n}\right)\cos nt. \tag{3}$$

Solution by the proposer. 1. Observe that the relation

$$a_1 \cos(\pi/2n) + \tfrac{1}{2}T(\pi+\pi/2n) + \tfrac{1}{2}T(\pi-\pi/2n) = 1 \tag{4}$$

holds for every $T(t)$ of the form (1). Now (1) and (4) imply that $a_1 \cos(\pi/2n) \leqq 1$, whence (2) follows.

2. Let us now assume that (2) holds with the equality sign. But then (4) and (1) imply that $T(\pi+\pi/2n)=T(\pi-\pi/2n)=0$, and therefore also that $T'(\pi+\pi/2n)=T'(\pi-\pi/2n)=0$. These four equations are easily shown to furnish (3) as the only solution. There remains to show that the trigonometric polynomial (3) is nonnegative for all real t, which is equivalent to showing that

$$\left(\cos\frac{\pi}{2n}\right)T(t+\pi) = -\cos t + \cos\frac{\pi}{2n} + \frac{1}{n}\sin\frac{\pi}{2n}\cos nt \geqq 0 \tag{5}$$

for all t. This fact is evident for t in the interval $[\frac{1}{2}\pi, \pi]$ and, since (5) is an even periodic function, there remains to consider only the case $0 \leqq t \leqq \frac{1}{2}\pi$.

Writing $\phi(t) = -\cos t + \cos(\pi/2n)$, $\psi(t) = (1/n)\sin \pi/2n \cos nt$, we observe:

(i) $\phi(\pi/2n) = \psi(\pi/2n) = 0$;

(ii) $\phi'(t) > |\psi'(t)|$ for $\pi/2n < t \leqq \pi/2$,

[because $\phi'(t) = \sin t > \sin(\pi/2n) \geqq |\sin(\pi/2n)\sin nt| = |\psi'(t)|$];

(iii) $\psi'(t) < 0$ and $|\psi'(t)| > \sin t = \phi'(t)$ for $0 < t < \pi/2n$,

[because $(\sin nt)/\sin t$ decreases over $[0, \pi/2n]$, so that

$$(\sin nt)/\sin t > (\sin n(\pi/2n))/\sin(\pi/2n)].$$

Finally, (i) and (ii) show that $\phi(t)+\psi(t)>0$ for $\pi/2n<t\leqq\pi/2$, also (i) and (iii) imply the same inequality for $0<t<\pi/2n$.

Also solved by L. Carlitz.

Convergent Sequence

5120 [1963, 673]. *Proposed by D. C. Olivier, Carleton College, Northfield, Minn.*

Define a sequence $\{v_n\}=\{v_n(x)\}$ recursively by $v_1=x$, $v_{n+1}=(2+1/n)v_n-1$, $n\geqq 1$. It is not hard to show that $\{v_n\}$ converges for at most one real value of x. Find x such that $\{v_n\}$ converges.

Solution by Roy O. Davies, The University, Leicester, England. By induction we have

$$v_n = 1\cdot 3\cdots(2n-1)(x-s_n)/(n-1)!, \qquad n=1,2,\cdots,$$

where $s_1=0$ and $s_{n+1}=s_n+[n!/1\cdot 3\cdots(2n+1)]$. If $\{v_n\}$ converges then $x-s_n\to 0$, whence

$$x=\lim s_n=\sum_{n=1}^{\infty}\frac{n!}{1\cdot 3\cdots(2n+1)}=\frac{1}{2}\pi-1.$$

(See, e.g., Bromwich, *Infinite Series*, 1st ed., p. 169.) For this x we find that

$$v_n=[n/(2n+1)]+[n(n+1)/(2n+1)(2n+3)]+\cdots,$$

and since the rth term here is increasing and tends to 2^{-r}, it follows that v_n indeed tends to $\frac{1}{2}+\frac{1}{4}+\cdots=1$.

Also solved by I. N. Baker, L. Carlitz, A. J. Casson, J. H. E. Cohn, H. D. Friedman, D. R. Hayes, Fulton Koehler, J. Koekoek, R. H. C. Newton, Ron Rietz, A. A. Sastry, H. Schwerdtfeger, D. W. Showalter, Arnold Singer, Robert Singleton, J. H. van Lint, K. H. van Weerden, J. Ernest Wilkins, Jr., Oswald Wyler, Max Wyman, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Lectures on Tensor Calculus and Differential Geometry. By Johan C. H. Gerretsen, P. Noordhoff N. V., Groningen, 1962. xii+204 pp. Dfl.25.

Here is an unusual introduction to the methods and principal results of the differential geometry of general manifolds. Frequently the development of this

An Elementary Introduction to the Theory of Probability. By B. V. Gnedenko and A. Ya. Khinchin. Translated from the fifth Russian edition by Leo F. Boron. Dover, New York, 1962. xii+130 pp. $1.45.

The high reputations of the authors and the fact that this is the fifth edition since 1945 both vouch for the quality of this small book. Part I, *Probabilities*, is a 54-page treatment of the standard rules for probability in finite sample spaces: addition, multiplication, conditional probabilities, independent events, Bayes' formula, the binomial (Bernoulli) distribution, and Bernoulli's theorem with Chebyshev's proof. (A minor correction: p. 51, line 7, read "distance more than" in place of "distance not more than.") The hypergeometric distribution is not included. Part II, *Random Variables*, is a 59-page treatment of random variables and their distribution laws, mean values, mean values of sums and products (of independent r.v.), mean deviation, standard deviation, probable deviation, Chebyshev's inequality and laws of large numbers, and normal distributions. The expected value operator is not used: instead, the authors use a bar to denote mean value. The *Conclusion*, 5 pages, sketches the development of probability theory from Fermat, Pascal, and Huygens to the present, with special attention to the contributions of Russians, but with credit also to the United States, France, Great Britain, Sweden, Japan, and Hungary. The discussion of the central limit theorem gives an excellent idea of why the normal distribution arises naturally in scientific applications.

The exposition is notably clear. New concepts are well motivated. Set language and symbolism are not used. The example-rule-example pattern is followed in almost every section. In general, no mathematics beyond high school algebra is required, though a few calculations use summation sigmas. There are no exercises, but the book would be an excellent supplement for elementary courses in probability in high school or college. The many examples illustrate how probability applies to a broad variety of practical and theoretical situations. It is unavoidable that, in such a brief treatment, some fine points will be ignored (for example, in connection with the probability distribution of the square or absolute value of a random variable which may assume negative and positive values), but the authors have succeeded admirably in presenting significant parts of probability at this level.

GEORGE B. THOMAS, JR., Massachusetts Institute of Technology

Linear Algebra and Matrix Theory. By E. D. Nering. Wiley, New York, 1963. xi+289 pp. $6.95.

This book should prove to be satisfactory as a text for a one semester course at an advanced undergraduate level. A rigorous treatment of most of the topics that normally make up a linear algebra course is given in the first five (of the six) chapters. These chapter headings are: I, Vector Spaces; II, Linear Transformations and Matrices; III, Determinants, Eigenvalues, and Similarity Transformations; IV, Linear Functionals, Bilinear Forms, Quadratic Forms; V, Orthogonal and Unitary Transformations, Normal Matrices. In the last chapter, Selected Applications of Linear Algebra, good but brief treatments of applications in fields such as linear programming and communication theory are given.

The author covers a good deal of material in a few pages. One omission that might bear mentioning is a discussion of the rational canonical forms for matrices under similarity. There is no development of field theory in this book. And no theory of polynomial forms is given. Knowledge of some of the elements of this theory is needed at various places, such as in the proof of the Hamilton-Cayley Theorem. In order to gain much insight into most of the applications discussed in the last chapter, a student would have to do a considerable amount of background reading. (A bibliography is given at the end of each section of this chapter.)

A good and ample supply of problems of various degrees of difficulty is found at the ends of the various sections of the chapters. Solutions or hints to solutions to many of these exercises are given at the end of the book.

P. W. CARRUTH, Swarthmore College

Solved and Unsolved Problems in Number Theory. Vol. I. By Daniel Shanks. Spartan Books, Baltimore, 1963. ix+229 pp. $7.50.

The title of this book is somewhat misleading. It is not a collection of problems, but a highly individualistic introductory textbook in number theory in which "problem—solution" is given preference over "theorem—proof." This is not to say that there are no theorems, but that the theorems are regarded not as end-products, but rather as stepping stones to the solutions of problems on which the author has already focused the reader's attention.

In the first chapter, for which the substratum is the problem of perfect numbers and Mersenne primes, one finds the unique factorization theorem, the theorems of Fermat and Euler, Euler's and Gauss's criteria and the law of quadratic reciprocity, all developed without mention of congruences, and interspersed with historical remarks, classical conjectures and much information on the results obtained by modern computers. In the second chapter congruences are introduced, and the group theoretic structure of the residue class groups is studied in much greater detail than is customary. The third chapter, built upon the Pythagorean theorem, ranges over Fermat's equation $x^n+y^n=z^n$ and its various elementary special cases, Pell's equation and continued fractions, and Lucas's criterion for primality of Mersenne numbers, with digressions on such matters as quantum physics, Pythagorean philosophy, and a purely arithmetic derivation of the Leibniz identity $\pi/4=1-1/3+1/5-\cdots$.

This is clearly not the book for a student who likes the orderly, polished and general (if not abstract) exposition to be found in most textbooks. On the other hand, the author has something to say, both philosophically and mathematically, which should be stimulating to students and enlightening even to professionals. No description of the contents can impart the flavor of the book; the interested reader is advised to examine a copy.

W. J. LEVEQUE, University of Michigan

arguments for a position are sometimes questionable in detail, e.g., using the early creativity of a few exceptional scientists to justify changing the structure of the entire public educational system, and sometimes wrong in basic concept, e.g., claiming that "mathematics is concerned with operations on abstract symbols," ignoring the fact that the operations act on mathematical objects. The author takes a bold stand in insisting that all ethical and moral imperative statements are predictions that the probability for survival of the individual and the species is decreased if a prohibited act is performed. This seems at once too nonempirical (often one can't compute the relevant possibilities, so one can't check the predictions) and too restrictive a definition to render the meaning of many such statements. As a general criticism, this reviewer thinks that the author, having found the single cause of all human behavior—survival the goal, and the scientific method the survival technique—is not sufficiently careful to check that his explanation accounts for all the facts. Finally, the book would serve its prime function better if it provided a bibliography for those readers who want to learn more about some of the sciences but aren't sure how to go about it.

R. C. MJOLSNESS, Los Alamos

Algebraic Logic. By P. R. Halmos. Chelsea, New York, 1962. 271 pp. $3.75.

By algebraic logic the author means that branch of general algebra which deals with algebraic structures mirroring in some sense certain formal logics. Examples of such structures, in historical order of investigation, are Boolean, Brouwerian, relation, projective, and cylindric algebras. Subsequent to the study of these algebras, Halmos introduced and investigated monadic and polyadic algebras, and the present monograph is the collection of all the articles Halmos has written on these two kinds of algebras. Polyadic algebras have algebraic operations mirroring sentential connectives, first-order quantifiers, and changes of variables, while monadic algebras are just a very special kind of polyadic algebras. The class of all polyadic algebras of a given degree is an equational class, and hence by any standard is a fitting object of study; but the polyadic algebras exactly corresponding to first-order logic are the locally finite ones of infinite degree which, unfortunately, do not form an equational class. The general polyadic algebras do mirror closely certain quite general logics with infinitely long expressions which are now being intensively studied.

As Halmos indicates, one of the main problems in algebraic logic is to state and prove algebraically various important theorems of logic. This was done by Tarski and later by Halmos for Gödel's completeness theorem. Halmos also treats algebraically, e.g., the description operator. The program has been carried through by Daigneault for Feferman-Vaught generalized products and for Craig's interpolation theorem. Notably still lacking is an algebraic treatment of Gödel's incompleteness theorem. It should also be mentioned that Daigneault and Keisler have generalized Halmos' treatment of the completeness theorem by showing that any simple polyadic algebra of infinite degree is isomorphic to an O-valued functional algebra. An open problem is to characterize those polyadic algebras of finite degree which are isomorphic to O-valued functional algebras; the reviewer has shown that not every simple algebra is of this kind.

Halmos' book is highly recommended as an introduction for those who wish to study logic from a purely algebraic point of view.

DONALD MONK, University of Colorado

An Introduction to the Calculus of Variations. By L. A. Pars. Wiley, New York, 1962. 350 pp. $8.50.

This book is somewhat similar to Bliss' Carus Monograph on the calculus of variations but is wider in scope. It was the author's intent to "give to the non-specialist a good insight into the fundamental ideas of the subject, a good working knowledge of the relevant techniques, and an adequate starting point for further study and research . . . ." It is also somewhat similar to Akhiezer's *Calculus of Variations* (English translation, Blaisdell Publishing Co., New York, 1962). There has been a need for books suitable for an introductory course in the calculus of variations. It is good to have another book as a possible choice for a text book in the subject.

The first five chapters are concerned with the ordinary problem in the plane. One chapter is devoted to concrete problems illustrating the theory. The multiplier rule for an isoperimetric problem in the plane is derived in the sixth chapter, and there is a brief discussion of the relation of the isoperimetric problem to Sturm-Liouville systems. There are many diagrams illustrating the geometric details of particular problems.

In chapter seven necessary conditions and the fundamental sufficiency theorem are given for the ordinary problem in three dimensions with an indication how to extend the theory to n-dimensions. It seems unnecessary to develop the theory first for the plane and then repeat the process for higher dimensions. It would have been simpler to treat the problem for n-dimensions from the start. Any objection a reader might have to this would be taken care of by the many concrete examples.

In chapter eight we find the multiplier rule for the Lagrange problem with n dependent and one independent variable. The next chapter deals with the parametric problem in the plane. The last chapter is entitled "Multiple Integrals." A more appropriate title might be "Dirichlet's Principle." Here some properties of harmonic functions are established. Dirichlet's principle is proved for a circular area and then, by following a method due to Poincaré, the principle is established for the general case. Since, as the author points out, a "higher standard of sophistication" is needed to follow a proof of Dirichlet's principle, it might have been preferable to use this sophistication on some theorems of Tonelli about direct methods in general.

There are thirty-five exercises at the end of the book that would keep a student very busy.

ALINE H. FRINK, Pennsylvania State University

Electronic Computers: Fundamentals, Systems, and Applications. Edited by Paul von Handel. Springer-Verlag, Vienna, and Prentice-Hall, Englewood Cliffs, N. J., 1961. 235 pp. $13.50.

The stated objective of this book is "to present an over-all view of various types of modern electronic computers" for "people . . . who have no special knowledge of computers." The main body of the work is by H. W. Gschwind, M. G. Jaenke and R. G. Tantzen of Holloman Air Force Base. There are chapters on digital computers, analog computers, digital differential analyzers and computing control systems.

In the chapter on digital computers such topics as over-all system organization, storage types, number systems, programming and applications are discussed. In the analog chapter one finds a discussion of presently available components, analog computer set-up and scaling. In these chapters there is a qualitative discussion of the error but no discussion of stability. In the case of the digital differential analyzer, an effort at error analysis is made but it does not seem to the reviewer to be adequate.

F. J. MURRAY, Duke University

Diophantine Approximations. By Ivan Niven. Wiley, New York, 1963. 68 pp. $5.00.

This self-contained monograph is an extension of the Hedrick lectures delivered by the author at the 1960 summer meeting of the MAA. It does not offer a complete survey of the field but is confined to the following topics: basic results on homogeneous and nonhomogeneous approximations of real numbers and analogous results for complex numbers; asymmetric approximation of irrational numbers; fundamental properties of the multiples of an irrational number, for both fractional and integral parts. A unique feature of this monograph is that continued fractions are not used. The inclusion of basic results for complex numbers is noteworthy, as well as the presence of many new proofs offered here for the first time. An attractive feature is the inclusion of a section entitled "Further results" at the end of each chapter to provide a bibliographic account of closely related work. The author's exposition is concise and lucid and the monograph will be extremely helpful and informative to specialist and nonspecialist alike.

W. E. BRIGGS, University of Colorado

BRIEF MENTION

Self-Organizing Systems. Edited by M. C. Yovits, G. T. Jacobi, and G. D. Goldstein. Spartan Books, Washington D. C., 1962. 563 pp. $12.00.

Proceedings of a 1962 conference of workers in several of the disciplines concerned with Self-Organizing Systems—Mathematics, Physics, Psychology, Biology, Embryology, Neurophysiology, etc.

Vector Analysis Including the Dynamics of a Rigid Body. By G. D. Smith. Oxford, New York, 1962. viii+192 pp. $4.00.

Via Vector to Tensor. By W. G. Bickley and R. E. Gibson, Wiley, New York, 1962. xvi+152 pp. $4.50.

These two texts are primarily for engineering students, and rely heavily on geometrical or physical motivations, arguments, and interpretations. The first includes five chapters on three-dimensional vector calculus, and concludes with a chapter on some simple problems in Newtonian mechanics. The second has a four-chapter "refresher course" in three-dimensional vector calculus followed by five chapters on the rudiments of tensor calculus, with some applications.

Fourier Series and Boundary Value Problems, 2nd ed. By Ruel V. Churchill. McGraw-Hill, New York, 1963. viii+248 pp. $6.75.

An extensive revision of the original 1941 edition of this well-known standard work.

Heat and its Workings. By Morton Mott-Smith. Dover, New York, 1962. 165 pp. $1.00.

First publication, 1933.

Technical Mathematics, 2nd ed. By H. S. Rice, and R. M. Knight. McGraw-Hill, New York, 1963. xiv+626 pp. $7.95.

Nonlinear Problems. Edited by Rudolph Langer. University of Wisconsin, 1963. xiii+321 pp. $7.50.

Proceedings of a Symposium conducted by the U. S. Army Research Center at Madison, Wisconsin, April 30–May 2, 1962.

Essential Business Mathematics, 4th ed. By L. R. Snyder. McGraw-Hill, New York, 1963. x+513 pp. $7.50.

An Introduction to Mathematical Probability. By J. L. Coolidge. Dover, New York, 1962. xii+214 pp. $1.35.

A republication of a classic which first appeared in 1925.

Principles of Mathematics, 2nd ed. By Carl Allendoerfer and Cletus Oakley. McGraw-Hill, New York, 1963. xi+540 pp. $7.95.

A considerably revised edition of this widely used text.

Complex Variable Theory and Transform Calculus with Technical Applications. 2nd ed. By N. W. McLachlan. Cambridge University Press, New York, 1963. xi+388 pp. $2.95.

Reprint of Second Edition, which appeared in 1953. Date of first edition, 1939.

Algebraic Curves. By Robert J. Walker. Dover, New York, 1962. x+215 pp. $1.60.

This is a republication of a well-known work first published in 1950.

Probabilities and Life. By Emile Borel. Translated from the French by Maurice Baudin. Dover, New York, 1963. vi+88 pp. $1.00.

This is a new English translation of the fourth French edition of a little book first published in 1943, in which the calculus of probabilities is applied to a number of questions which relate either to everyday living or to illness and death.

Regular Polytopes. By H. M. S. Coxeter. Macmillan, New York, 1963. xix+321 pp. $4.50.

This is the second edition of a book which first appeared in 1948.

Antiplane Elastic Systems. Ergebnisse der Angewandten Mathematik, No. 8. By L. M. Milne-Thompson. Academic Press, New York, 1963. vii+265 pp. $11.00.

Mathematical Theory of Elastic Equilibrium (Recent results). Ergebnisse der Angewandten Mathematik, No. 7. By Giuseppe Grioli. Academic Press, New York, 1962. viii+167 pp. $5.50.

Mathematical Optimization Techniques. By Richard Bellman, editor. University of California Press, 1963. xii+346 pp. $8.50.

Papers presented at the Symposium on Mathematical Optimization Techniques, Santa Monica, October 1960. The book is divided into four parts: I. Aircraft, Rockets, and Guidance; II. Communication, Prediction, and Decision; III. Programming, Combinatorics, and Design; IV. Models, Automation, and Control.

Angles and In- and Ex-Elements of Triangles and Tetrahedra. By Kesiraju Satyanarayana. Bangalore Press, Bangalore City, 1962. xiii+135 pp. 5 Rupees.

Journal of Research in Science Teaching, vol. 1, Issue 1. J. Stanley Marshall, editor. Wiley, New York, 1963. 98 pp.

Introductory Statistical Mechanics for Physicists. By D. K. C. MacDonald. Wiley, New York, 1963. ix+176 pp. $6.75.

Elementary "applied" statistical mechanics with emphasis on solid state phenomena.

Quantum Mechanics for Mathematicians and Physicists. By Ernest Ikenberry. Oxford University Press, New York, 1962. 269 pp. $8.00.

A fresh concise introduction to the elements of the theory emphasizing the mathematical aspects of its development. A useful section-by-section bibliography is included.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor H. L. Alder, University of California, Davis, represented the Association at the Convocation held as part of the Dedication of the California State College at Hayward on May 2, 1964.

Professor R. R. Stoll, Oberlin College, represented the Association at the dedication of the Charles F. Kettering Science Center at Ashland College on March 14, 1964.

Brigham Young University: Assistant Professors K. L. Hillam and L. J. Olpin have been promoted to Associate Professors; Mr. H. E. Wickes has been promoted to Assistant Professor; Assistant Professor H. G. Moore has been granted a leave of absence and awarded an NSF Science Faculty Fellowship for study at the University of California at Santa Barbara.

Assistant Professor L. A. Fiedler, Black Hawk College, has been promoted to Associate Professor and appointed Acting Head of the Mathematics Department.

Professor Karl Menger, Illinois Institute of Technology, has been appointed Visiting Professor for the spring semester at the University of Arizona.

Associate Professor Gloria Olive, Anderson College, has been promoted to Professor and Chairman of the Mathematics Department.

Professor Emeritus L. K. Adkins, Wisconsin State College, LaCrosse, died on November 11, 1963. He was a charter member of the Association.

Professor Emeritus J. E. Dotterer, Manchester College, died on January 21, 1964. He was a charter member of the Association.

Assistant Professor Corinne R. Hattan, University of Illinois, died on February 7, 1964. She was a member of the Association for 26 years.

Professor S. S. Wilks, Princeton University, died on March 8, 1964. He was a member of the Association for 23 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the United States Naval Academy, Annapolis, Maryland, on April 27, 1963. Professor Herta T. Freitag, Chairman of the section, presided. The invited address on "Men and Mathematics: a Plea for the Historical Sense in Mathematics," was delivered by Dr. Philip J. Davis, Chief Numerical Analysis Division, National Bureau of Standards, Washington, D. C., the recipient of the Chauvenet Prize, 1963.

At the business meeting the following officers were elected: Chairman, Dr. John W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.; Vice Chairmen, Professor George H. Butcher, Howard University, Washington, D. C. and Professor Raymond W. Moller, Catholic University of America, Washington, D. C.; Secretary, Professor Samuel S. Saslaw, U. S. Naval Academy, Annapolis, Maryland; Treasurer, Professor Stanley B. Jackson, University of Maryland, College Park, Maryland.

The following program was presented:

1. *Tailgater, a simultaneous compiler*, by Professor H. Kaplan, U. S. Naval Academy.

2. *A theory of primes and Cramer's conjecture*, by Commander F. B. Correia, USN, U. S. Nava Academy.

3. *Convex metrics*, by Dr. Christoph Witzgall, National Bureau of Standards, Washington, D. C.

4. *Error analysis of the magnetic attitude prediction program for the Tiros satellites*, by W. H. Land, Jr., I.B.M. Corporation, Bethesda, Maryland.

5. *A least squares unit vector perpendicular to a given set of vectors*, by H. E. Castro, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

6. *Ship location by means of an acoustic range*, by Professor R. P. Bailey, U. S. Naval Academy.

7. *A theorem on convex programming*, by Dr. A. J. Goldman, National Bureau of Standards, Washington, D. C.

S. S. Saslaw, *Secretary*

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at American University, Washington, D. C. on Saturday, December 14, 1963. Dr. John W. Wrench, Jr., Chairman of the section, presided. Dr. F. Joachim Weyl, Deputy Chief and Chief Scientist, gave the invited address on "Elementary